Banach space mixed formulations for the convection-diffusion-reaction system based upon Brinkman–Forchheimer equations

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Talca Numérica I

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The model problem

$\Omega \subset \mathrm{R}^n$, $n \in \{2,3\}$, bounded domain with polyhedral boundary Γ.

$$
\begin{aligned}
-\,\mathbf{div}\big(\nu\,\nabla \mathbf{u}\big) \,+\, \mathtt{D}\,\mathbf{u} \,+\, \mathtt{F}\,|\mathbf{u}|^{\rho-2}\,\mathbf{u} \,+\, \nabla p \,&=\, \mathbf{f}(\phi) && \text{in} \quad \Omega\,,\\
\mathrm{div}(\mathbf{u}) \,&=\, f && \text{in} \quad \Omega\,,\\
-\kappa\,\Delta\,\phi \,+\, \mathbf{u}\cdot\nabla\phi \,+\, \eta\,\phi \,&=\, g && \text{in} \quad \Omega\,,\\
\mathbf{u} \,=\, \mathbf{u}_{\mathrm{D}} \quad \text{and} \quad \phi \,&=\, \phi_{\mathrm{D}} && \text{on} \quad \Gamma\,,
\end{aligned}
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\mathrm{div}(\mathbf{u}) \,&=\, f \qquad \text{in} \quad \Omega\,, \\
-\kappa\,\Delta\,\phi \,+\, \mathbf{u}\cdot\nabla\phi \,+\, \eta\,\phi \,&=\, g \qquad \text{in} \quad \Omega\,, \\
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\end{aligned}
$$

Data assumptions

$\nu > 0$, $\mathtt{D} > 0$, $\mathtt{F} > 0$, $\rho \in [3,4]$, $\kappa > 0$ and $\eta > 0$.

$$
\nu_0 \,\leqslant\, \nu(\mathbf{x}) \,\leqslant\, \nu_1\,, \quad \mathtt{D}_0 \,\leqslant\, \mathtt{D}(\mathbf{x}) \,\leqslant\, \mathtt{D}_1\,, \quad \text{and} \quad \mathtt{F}_0 \,\leqslant\, \mathtt{F}(\mathbf{x}) \,\leqslant\, \mathtt{F}_1\,,
$$

$$
\mathbf{f}(\phi) \,:=\, -(\phi - \phi_{\mathtt{r}})\,\mathbf{g}
$$

Figure 1: A *completo mojado* can be regarded as a highly porous medium: the sauce constitutes the fluid phase, while its dissolved ingredients serve as solutes.

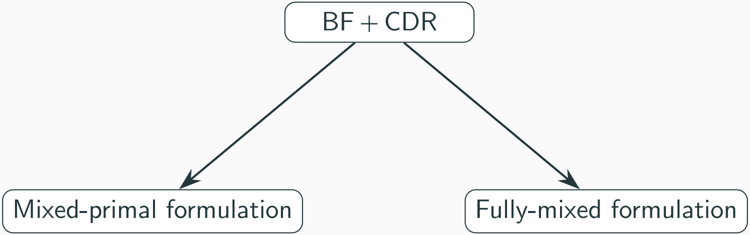
Compatibility condition

$$\int_\Gamma \mathbf{u}_\mathrm{D} \cdot \mathbf{n} \; = \; \int_\Omega f \, .$$

Uniqueness of the pressure

$$p \in \mathrm{L}^2_0(\Omega) \; := \; \left\{ q \in \mathrm{L}^2(\Omega) : \quad \int_\Omega q = 0 \right\} \, .$$

What do we want to do?



What are the advantages and disadvantages of each formulation?

The mixed-primal approach

Introducing the pseudostress tensor $\boldsymbol{\sigma}$:

$$\boldsymbol{\sigma} \;:=\; \nu\,\nabla\mathbf{u} \,-\, p\,\mathbb{I} \quad \text{in} \quad \Omega\,.$$

$$\Longrightarrow\; p = -\frac{1}{n}\mathrm{tr}(\boldsymbol{\sigma}) + \frac{\nu}{n}\,f \quad \text{and} \quad \frac{1}{\nu}\,\boldsymbol{\sigma}^{\mathrm{d}} = \nabla\mathbf{u} - \frac{1}{n}\,f\,\mathbb{I}.$$

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Find $\mathbf{u}$, $\boldsymbol{\sigma}$ and ϕ such that:

$$\begin{aligned}
-\mathbf{div}\big(\boldsymbol{\sigma}\big) \;+\; \mathtt{D}\,\mathbf{u} \;+\; \mathtt{F}\,|\mathbf{u}|^{\rho-2}\,\mathbf{u} \;&=\; \mathbf{f}(\phi) && \text{in} \quad \Omega\\
\frac{1}{\nu}\,\boldsymbol{\sigma}^{\mathrm{d}} \;&=\; \nabla\mathbf{u} \;-\; \frac{1}{n}\,f\,\mathbb{I} && \text{in} \quad \Omega\\
-\kappa\,\Delta\,\phi \;+\; \mathbf{u}\cdot\nabla\phi \;+\; \eta\,\phi \;&=\; g && \text{in} \quad \Omega\\
\mathbf{u} \;=\; \mathbf{u}_{\mathrm{D}}\,, \quad \phi \;&=\; \phi_{\mathrm{D}} && \text{on} \quad \Gamma\\
\int_\Omega \Big\{\mathrm{tr}(\boldsymbol{\sigma}) - \nu\,f\Big\} \;&=\; 0.
\end{aligned}$$

Weak formulation

Multiplying by a vector field $\mathbf{v}$ and a tensor field $\boldsymbol{\tau}$ in the BF equations, and integrating by parts in the constitutive equation:

$$\int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\sigma}) - \int_\Omega \mathtt{D}\, \mathbf{u} \cdot \mathbf{v} - \int_\Omega \mathtt{F}\, |\mathbf{u}|^{\rho-2} \mathbf{u} \cdot \mathbf{v} = - \int_\Omega \mathbf{f}(\phi) \cdot \mathbf{v}.$$

$$\int_\Omega \frac{1}{\nu} \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \int_\Omega \mathbf{u} \cdot \mathbf{div}(\boldsymbol{\tau}) = \langle \boldsymbol{\tau}\, \mathbf{n}, \mathbf{u}_{\mathrm{D}} \rangle - \frac{1}{n} \int_\Omega f\, \mathrm{tr}(\boldsymbol{\tau}).$$

Bounding the Forchheimer term:

$$\left| \int_\Omega \mathtt{F}\, |\mathbf{z}|^{\rho-2} \mathbf{u} \cdot \mathbf{v} \right| \leqslant \mathtt{F}_1 \left\{ \int_\Omega |\mathbf{z}|^{\ell(\rho-2)} |\mathbf{u}|^{\ell} \right\}^{1/\ell} \|\mathbf{v}\|_{0,j;\Omega}$$

$$\leqslant \mathtt{F}_1 \|\mathbf{z}\|^{\rho-2}_{0,\ell(\rho-1);\Omega} \|\mathbf{u}\|_{0,\ell(\rho-1);\Omega} \|\mathbf{v}\|_{0,j;\Omega},$$

$$j, \ell \in (1, +\infty), \qquad 1/j + 1/\ell = 1$$

$$\left| \int_\Omega \mathtt{F} \, |\mathbf{z}|^{\rho-2} \, \mathbf{u} \cdot \mathbf{v} \right| \leqslant \mathtt{F}_1 \, \|\mathbf{z}\|^{\rho-2}_{0,\ell(\rho-1);\,\Omega} \, \|\mathbf{u}\|_{0,\ell(\rho-1);\,\Omega} \, \|\mathbf{v}\|_{0,j;\,\Omega}$$

Suppose $\ell(\rho-1) = j$. Then $j = \rho$,

$$\left| \int_\Omega \mathtt{F} \, |\mathbf{z}|^{\rho-2} \, \mathbf{u} \cdot \mathbf{v} \right| \leqslant \mathtt{F}_1 \, \|\mathbf{z}\|^{\rho-2}_{0,\rho;\,\Omega} \, \|\mathbf{u}\|_{0,\rho;\,\Omega} \, \|\mathbf{v}\|_{0,\rho;\,\Omega}$$

We require:

$$\mathbf{u}, \mathbf{v}, \mathbf{z} \in \mathbf{L}^\rho(\Omega).$$

So far ...

Find $\mathbf{u} \in \mathbf{L}^\rho(\Omega)$ such that

$$\begin{aligned}
&\int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\sigma}) - \int_\Omega \mathtt{D}\,\mathbf{u} \cdot \mathbf{v} - \int_\Omega \mathtt{F}\,|\mathbf{u}|^{\rho-2}\,\mathbf{u} \cdot \mathbf{v} = -\int_\Omega \mathbf{f}(\phi) \cdot \mathbf{v} \quad \forall\, \mathbf{v} \in \mathbf{L}^\rho(\Omega)\\
&\int_\Omega \frac{1}{\nu}\,\boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \int_\Omega \mathbf{u} \cdot \mathbf{div}(\boldsymbol{\tau}) = \langle \boldsymbol{\tau}\,\mathbf{n}, \mathbf{u}_{\mathrm{D}} \rangle - \frac{1}{n} \int_\Omega f\,\mathrm{tr}(\boldsymbol{\tau}).
\end{aligned}$$

So far ...

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\end{aligned}
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$$\boxed{\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_\ell; \Omega)}$$

So far . . .

Find $\mathbf{u} \in \mathbf{L}^\rho(\Omega)$ such that

$$\begin{aligned}
&\int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\sigma}) - \int_\Omega \mathtt{D}\, \mathbf{u} \cdot \mathbf{v} - \int_\Omega \mathtt{F}\, |\mathbf{u}|^{\rho-2}\, \mathbf{u} \cdot \mathbf{v} = -\int_\Omega \mathbf{f}(\phi) \cdot \mathbf{v} \quad \forall\, \mathbf{v} \in \mathbf{L}^\rho(\Omega)\\
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Find $\mathbf{u} \in \mathbf{L}^\rho(\Omega)$ and $\boldsymbol{\sigma} \in \mathbb{H}(\mathbf{div}_\ell; \Omega)$ such that

$$\begin{aligned}
&\int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\sigma}) - \int_\Omega \mathtt{D}\, \mathbf{u} \cdot \mathbf{v} - \int_\Omega \mathtt{F}\, |\mathbf{u}|^{\rho-2}\, \mathbf{u} \cdot \mathbf{v} = -\int_\Omega \mathbf{f}(\phi) \cdot \mathbf{v} \quad \forall\, \mathbf{v} \in \mathbf{L}^\rho(\Omega),\\
&\int_\Omega \frac{1}{\nu}\, \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \int_\Omega \mathbf{u} \cdot \mathbf{div}(\boldsymbol{\tau}) = \langle \boldsymbol{\tau}\, \mathbf{n}, \mathbf{u}_{\mathrm{D}} \rangle - \frac{1}{n} \int_\Omega f\, \mathrm{tr}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_\ell; \Omega).
\end{aligned}$$

Bounding RHS: (recall $\mathbf{f}(\phi) = -(\phi - \phi_{\mathtt{r}})\,\mathbf{g}$)

$$\left|\int_\Omega \mathbf{f}(\phi)\cdot\mathbf{v}\right| \leqslant \|\phi - \phi_{\mathtt{r}}\|_{0,\frac{2\rho}{(\rho-2)};\Omega}\,\|\mathbf{g}\|_{0,\Omega}\,\|\mathbf{v}\|_{0,\rho;\Omega}$$
$$\leqslant \left(\|\phi\|_{0,s;\Omega} + \|\phi_{\mathtt{r}}\|_{0,s;\Omega}\right)\|\mathbf{g}\|_{0,\Omega}\,\|\mathbf{v}\|_{0,\rho;\Omega},$$

where $s := 2\rho/(\rho-2) \in [4,6]$.

$$\left|\int_\Omega \mathbf{f}(\phi)\cdot\mathbf{v}\right| \leqslant \left(\|i_s\|\,\|\phi\|_{1,\Omega} + \|\phi_{\mathtt{r}}\|_{0,s;\Omega}\right)\|\mathbf{g}\|_{0,\Omega}\,\|\mathbf{v}\|_{0,\rho;\Omega}$$

Here, $i_s : \mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^s(\Omega)$.

$$\boxed{\phi \in \mathrm{H}^1(\Omega)}$$

So far ...

Find $\mathbf{u} \in \mathbf{L}^\rho(\Omega)$, $\boldsymbol{\sigma} \in \mathbb{H}(\mathbf{div}_\ell;\Omega)$ and $\phi \in \mathrm{H}^1(\Omega)$ such that

$$\begin{aligned}
&\int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\sigma}) - \int_\Omega \mathtt{D}\,\mathbf{u}\cdot\mathbf{v} - \int_\Omega \mathtt{F}\,|\mathbf{u}|^{\rho-2}\,\mathbf{u}\cdot\mathbf{v} = -\int_\Omega \mathbf{f}(\phi)\cdot\mathbf{v} \quad \forall\,\mathbf{v}\in\mathbf{L}^\rho(\Omega),\\
&\int_\Omega \frac{1}{\nu}\,\boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) = \langle \boldsymbol{\tau}\,\mathbf{n}, \mathbf{u}_{\mathrm{D}}\rangle - \frac{1}{n}\int_\Omega f\,\mathrm{tr}(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_\ell;\Omega),
\end{aligned}$$

with data $\phi_{\mathbf{r}} \in \mathrm{L}^s(\Omega)$, $\mathbf{u}_{\mathrm{D}} \in \mathbf{H}^{1/2}(\Gamma)$ and $f \in \mathrm{L}^2(\Omega)$.

Restrict to $\mathbb{H}_0(\mathbf{div}_\ell;\Omega)$

Recall that...

1. $\mathbb{H}_0(\mathbf{div}_\ell;\Omega) := \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_\ell;\Omega) : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = 0 \right\}$,
2. $\mathbb{H}(\mathbf{div}_\ell;\Omega) = \mathbb{H}_0(\mathbf{div}_\ell;\Omega) \oplus \mathrm{R}\,\mathbb{I}$.

Our unknown $\boldsymbol{\sigma} \in \mathbb{H}(\mathbf{div}_\ell;\Omega)$ can be decomposed as $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + d_{\boldsymbol{\sigma}}\,\mathbb{I}$.

Uniqueness condition for the pressure says $\int_\Omega (\mathrm{tr}(\boldsymbol{\sigma}) - \nu\, f) = 0$, then

$$d_{\boldsymbol{\sigma}} = \frac{1}{n\,|\Omega|} \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) = \frac{1}{n\,|\Omega|} \int_\Omega \nu\, f.$$

We can seek $\boldsymbol{\sigma}_0$ instead of $\boldsymbol{\sigma}$!

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We can seek $\boldsymbol{\sigma}_0$ instead of $\boldsymbol{\sigma}$! Notation: $\boldsymbol{\sigma} \leftarrow \boldsymbol{\sigma}_0$

Restrict to $\mathbb{H}_0(\mathbf{div}_\ell;\Omega)$

Moreover ...

$$\int_\Omega \frac{1}{\nu}\,\boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) = \langle \boldsymbol{\tau}\,\mathbf{n}, \mathbf{u}_{\mathrm{D}}\rangle - \frac{1}{n}\int_\Omega f\,\mathrm{tr}(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_\ell;\Omega)$$

$$\Longleftrightarrow$$

$$\int_\Omega \frac{1}{\nu}\,\boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) = \langle \boldsymbol{\tau}\,\mathbf{n}, \mathbf{u}_{\mathrm{D}}\rangle - \frac{1}{n}\int_\Omega f\,\mathrm{tr}(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_\ell;\Omega)$$

So far . . .

Find $\mathbf{u} \in \mathbf{L}^\rho(\Omega)$, $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div}_\ell; \Omega)$ and $\phi \in \mathrm{H}^1(\Omega)$ such that

$$\begin{aligned}
&\int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\sigma}) - \int_\Omega \mathtt{D}\, \mathbf{u} \cdot \mathbf{v} - \int_\Omega \mathtt{F}\, |\mathbf{u}|^{\rho-2} \mathbf{u} \cdot \mathbf{v} = - \int_\Omega \mathbf{f}(\phi) \cdot \mathbf{v} \quad \forall\, \mathbf{v} \in \mathbf{L}^\rho(\Omega), \\
&\int_\Omega \frac{1}{\nu} \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \int_\Omega \mathbf{u} \cdot \mathbf{div}(\boldsymbol{\tau}) = \langle \boldsymbol{\tau}\, \mathbf{n}, \mathbf{u}_{\mathrm{D}} \rangle - \frac{1}{n} \int_\Omega f \,\mathrm{tr}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_\ell; \Omega),
\end{aligned}$$

with data $\phi_{\mathrm{r}} \in \mathrm{L}^s(\Omega)$, $\mathbf{u}_{\mathrm{D}} \in \mathbf{H}^{1/2}(\Gamma)$ and $f \in \mathrm{L}^2(\Omega)$.

What about the convection-diffusion-reaction equation?

CDR: $\quad -\kappa\,\Delta\,\phi + \mathbf{u}\cdot\nabla\phi + \eta\,\phi = g \quad \text{in} \quad \Omega.$

Introduce $\lambda := -\kappa\,\nabla\phi\cdot\mathbf{n} \in \mathrm{H}^{-1/2}(\Gamma)$, test against $\psi \in \mathrm{H}^1(\Omega)$, integrate by parts:

$$\kappa\int_\Omega \nabla\phi\cdot\nabla\psi + \int_\Omega(\mathbf{u}\cdot\nabla\phi)\,\psi + \eta\int_\Omega \phi\,\psi + \langle\lambda,\psi\rangle_\Gamma = \int_\Omega g\,\psi \qquad \forall\,\psi\in\mathrm{H}^1(\Omega),$$

with the datum $g \in \mathrm{L}^2(\Omega)$.

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$$\kappa\int_\Omega \nabla\phi\cdot\nabla\psi + \int_\Omega(\mathbf{u}\cdot\nabla\phi)\,\psi + \eta\int_\Omega \phi\,\psi + \langle\lambda,\psi\rangle_\Gamma = \int_\Omega g\,\psi \qquad \forall\,\psi\in\mathrm{H}^1(\Omega),$$

with the datum $g \in \mathrm{L}^2(\Omega)$.

Dirichlet condition is imposed weakly via

$$\langle\xi,\phi\rangle_\Gamma = \langle\xi,\phi_{\mathrm{D}}\rangle_\Gamma \qquad \forall\,\xi\in\mathrm{H}^{-1/2}(\Gamma).$$

BF coupled with CDR — Mixed-primal formulation

Find $\mathbf{u} \in \mathbf{L}^\rho(\Omega)$, $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div}_\ell;\Omega)$, $\phi \in \mathrm{H}^1(\Omega)$ and $\lambda \in \mathrm{H}^{-1/2}(\Gamma)$ such that

$$
\begin{aligned}
&\int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\sigma}) - \int_\Omega \mathtt{D}\,\mathbf{u}\cdot\mathbf{v} - \int_\Omega \mathtt{F}\,|\mathbf{u}|^{\rho-2}\,\mathbf{u}\cdot\mathbf{v} = -\int_\Omega \mathbf{f}(\phi)\cdot\mathbf{v} && \forall\, \mathbf{v} \in \mathbf{L}^\rho(\Omega),\\
&\int_\Omega \frac{1}{\nu}\,\boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) = \langle \boldsymbol{\tau}\,\mathbf{n}, \mathbf{u}_{\mathrm{D}}\rangle - \frac{1}{n}\int_\Omega f\,\mathrm{tr}(\boldsymbol{\tau}) && \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_\ell;\Omega),\\
&\kappa \int_\Omega \nabla\phi\cdot\nabla\psi + \int_\Omega (\mathbf{u}\cdot\nabla\phi)\,\psi + \eta\int_\Omega \phi\,\psi + \langle\lambda,\psi\rangle_\Gamma = \int_\Omega g\,\psi && \forall\, \psi \in \mathrm{H}^1(\Omega),\\
&\langle\xi,\phi\rangle_\Gamma = \langle\xi,\phi_{\mathrm{D}}\rangle_\Gamma && \forall\, \xi \in \mathrm{H}^{-1/2}(\Gamma).
\end{aligned}
$$

BF coupled with CDR — Mixed-primal formulation

Find $(\boldsymbol{\sigma}, \mathbf{u}, \phi, \lambda) \in \mathbb{H}_0(\mathbf{div}_\ell; \Omega) \times \mathbf{L}^\rho(\Omega) \times \mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma)$ such that

$$
\begin{array}{llll}
\displaystyle\int_\Omega \frac{1}{\nu} \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} & \displaystyle + \int_\Omega \mathbf{u} \cdot \mathbf{div}(\boldsymbol{\tau}) & = & \displaystyle \langle \boldsymbol{\tau}\, \mathbf{n}, \mathbf{u}_{\mathrm{D}} \rangle - \frac{1}{n} \int_\Omega f \, \mathrm{tr}(\boldsymbol{\tau}), \\
\displaystyle\int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\sigma}) & \displaystyle - \int_\Omega \mathtt{D}\, \mathbf{u} \cdot \mathbf{v} - \int_\Omega \mathtt{F}\, |\mathbf{u}|^{\rho-2} \mathbf{u} \cdot \mathbf{v} & = & \displaystyle - \int_\Omega \mathbf{f}(\phi) \cdot \mathbf{v}, \\
\displaystyle \kappa \int_\Omega \nabla\phi \cdot \nabla\psi + \int_\Omega (\mathbf{u} \cdot \nabla\phi)\, \psi + \eta \int_\Omega \phi\, \psi & + \langle \lambda, \psi \rangle_\Gamma & = & \displaystyle \int_\Omega g\, \psi, \\
\langle \xi, \phi \rangle_\Gamma & & = & \langle \xi, \phi_{\mathrm{D}} \rangle_\Gamma,
\end{array}
$$

$$\forall\, (\boldsymbol{\tau}, \mathbf{v}, \psi, \xi) \in \mathbb{H}_0(\mathbf{div}_\ell; \Omega) \times \mathbf{L}^\rho(\Omega) \times \mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma).$$

BF coupled with CDR — Mixed-primal formulation

Find $(\boldsymbol{\sigma}, \mathbf{u}, \phi, \lambda) \in \mathbb{H}_0(\mathbf{div}_\ell;\Omega) \times \mathbf{L}^\rho(\Omega) \times \mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma)$ such that

$$\begin{aligned}
&\int_\Omega \frac{1}{\nu}\,\boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} && + \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) && = && \langle \boldsymbol{\tau}\,\mathbf{n}, \mathbf{u}_{\mathrm{D}}\rangle - \frac{1}{n}\int_\Omega f\,\mathrm{tr}(\boldsymbol{\tau}),\\
&\int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) && - \int_\Omega \mathtt{D}\,\mathbf{u}\cdot\mathbf{v} - \underbrace{\int_\Omega \mathtt{F}\,|\mathbf{u}|^{\rho-2}\,\mathbf{u}\cdot\mathbf{v}}_{\text{nonlinear}} && = && -\int_\Omega \mathbf{f}(\phi)\cdot\mathbf{v},\\
&\kappa\int_\Omega \nabla\phi\cdot\nabla\psi + \underbrace{\int_\Omega (\mathbf{u}\cdot\nabla\phi)\,\psi}_{\text{nonlinear}} + \eta\int_\Omega \phi\,\psi && + \langle \lambda, \psi\rangle_\Gamma && = && \int_\Omega g\,\psi,\\
&\langle \xi, \phi\rangle_\Gamma && && = && \langle \xi, \phi_{\mathrm{D}}\rangle_\Gamma,
\end{aligned}$$

$$\forall\,(\boldsymbol{\tau}, \mathbf{v}, \psi, \xi) \in \mathbb{H}_0(\mathbf{div}_\ell;\Omega) \times \mathbf{L}^\rho(\Omega) \times \mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma).$$

Some definitions

$\mathcal{H} := \mathbb{H}_0(\mathbf{div}_\ell;\Omega)$ and $\mathcal{Q} := \mathbf{L}^\rho(\Omega)$. For each $(\mathbf{z},\varphi) \in \mathcal{Q}\times \mathrm{H}^1(\Omega)$:

$\mathbf{a} : \mathcal{H}\times\mathcal{H} \to \mathrm{R}$, $\mathbf{b} : \mathcal{H}\times\mathcal{Q}\to \mathrm{R}$, $\mathbf{c}_{\mathbf{z}} : \mathcal{Q}\times\mathcal{Q}\to\mathrm{R}$, $\mathbf{F} : \mathcal{H}\to\mathrm{R}$ and $\mathbf{G}_\varphi : \mathcal{Q}\to\mathrm{R}$,

$$\mathbf{a}(\boldsymbol{\chi},\boldsymbol{\tau}) := \int_\Omega \frac{1}{\nu}\boldsymbol{\chi}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}}, \qquad \mathbf{b}(\boldsymbol{\tau},\mathbf{v}) := \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau}),$$

$$\mathbf{c}_{\mathbf{z}}(\mathbf{w},\mathbf{v}) := \int_\Omega \mathrm{D}\,\mathbf{w}\cdot\mathbf{v} + \int_\Omega \mathrm{F}\,|\mathbf{z}|^{\rho-2}\mathbf{w}\cdot\mathbf{v},$$

$$\mathbf{F}(\boldsymbol{\tau}) := \langle\boldsymbol{\tau}\,\mathbf{n},\mathbf{u}_{\mathrm{D}}\rangle - \frac{1}{n}\int_\Omega f\,\mathrm{tr}(\boldsymbol{\tau}), \quad \mathbf{G}_\varphi(\mathbf{v}) := -\int_\Omega \mathbf{f}(\varphi)\cdot\mathbf{v},$$

$a_{\mathbf{z}} : \mathrm{H}^1(\Omega)\times\mathrm{H}^1(\Omega)\to\mathrm{R}$, $b : \mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma)\to\mathrm{R}$, $\mathrm{F} : \mathrm{H}^1(\Omega)\to\mathrm{R}$ and $\mathrm{G} : \mathrm{H}^{-1/2}(\Gamma)\to\mathrm{R}$,

$$a_{\mathbf{z}}(\zeta,\psi) := \kappa\int_\Omega \nabla\zeta\cdot\nabla\psi + \int_\Omega (\mathbf{z}\cdot\nabla\zeta)\,\psi + \eta\int_\Omega \zeta\,\psi,$$

$$b(\psi,\xi) := \langle\xi,\psi\rangle_\Gamma, \quad \mathrm{F}(\psi) := \int_\Omega g\,\psi \quad \text{and} \quad \mathrm{G}(\xi) := \langle\xi,\phi_{\mathrm{D}}\rangle_\Gamma.$$

BF coupled with CDR — Mixed-primal formulation

Find $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathcal{H} \times \mathcal{Q}$ and $(\phi, \lambda) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma)$ such that

$$
\begin{aligned}
\mathbf{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau}, \mathbf{u}) &= \mathbf{F}(\boldsymbol{\tau}) && \forall\, \boldsymbol{\tau} \in \mathcal{H}\,,\\
\mathbf{b}(\boldsymbol{\sigma}, \mathbf{v}) - \mathbf{c}_{\mathbf{u}}(\mathbf{u}, \mathbf{v}) &= \mathbf{G}_{\phi}(\mathbf{v}) && \forall\, \mathbf{v} \in \mathcal{Q}\,,\\
a_{\mathbf{u}}(\phi, \psi) + b(\psi, \lambda) &= \mathrm{F}(\psi) && \forall\, \psi \in \mathrm{H}^1(\Omega)\,,\\
b(\phi, \xi) &= \mathrm{G}(\xi) && \forall\, \xi \in \mathrm{H}^{-1/2}(\Gamma)\,,
\end{aligned}
$$

A fixed-point strategy

Uncoupled BF

$\mathbf{S} : \mathcal{Q} \times \mathrm{H}^1(\Omega) \to \mathcal{Q}$ such that $\mathbf{S}(\mathbf{z}, \varphi) := \mathbf{u}$ where $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathcal{H} \times \mathcal{Q}$ denotes the unique solution of

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau}, \mathbf{u}) &= \mathbf{F}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathcal{H}\,, \\ \mathbf{b}(\boldsymbol{\sigma}, \mathbf{v}) - \mathbf{c}_{\mathbf{z}}(\mathbf{u}, \mathbf{v}) &= \mathbf{G}_{\varphi}(\mathbf{v}) \qquad \forall\, \mathbf{v} \in \mathcal{Q}\,. \end{aligned}$$

Uncoupled CDR

$\tilde{\mathbf{S}} : \mathcal{Q} \to \mathrm{H}^1(\Omega)$ such that $\tilde{\mathbf{S}}(\mathbf{z}) := \phi$, where $(\phi, \lambda) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma)$ is the unique solution of

$$\begin{aligned} a_{\mathbf{z}}(\phi, \psi) + b(\psi, \lambda) &= \mathrm{F}(\psi) \qquad \forall\, \psi \in \mathrm{H}^1(\Omega)\,, \\ b(\phi, \xi) &= \mathrm{G}(\xi) \qquad \forall\, \xi \in \mathrm{H}^{-1/2}(\Gamma), \end{aligned}$$

Global fixed-point operator

$\mathbf{T} : \mathcal{Q} \to \mathcal{Q}$ defined by

$$\mathbf{T}(\mathbf{z}) := \mathbf{S}(\mathbf{z}, \widetilde{\mathbf{S}}(\mathbf{z})) \qquad \forall\, \mathbf{z} \in \mathcal{Q}.$$

Well-posedness of BF

We shall use the next result:

Theorem[1]

Let H and Q be reflexive Banach spaces, and let $a : H \times H \to \mathrm{R}$, $b : H \times Q \to \mathrm{R}$, and $c : Q \times Q \to \mathrm{R}$ be given bounded bilinear forms. Let V be the kernel of $H \ni \tau \mapsto b(\tau, \cdot) \in Q'$. Assume that

1. a and c are symmetric and positive semi-definite.
2. $\exists\, \alpha > 0$ such that $\displaystyle\sup_{0 \neq \tau \in V} \frac{a(\vartheta, \tau)}{\|\tau\|_H} \geqslant \alpha \|\vartheta\|_H \qquad \forall\, \vartheta \in V.$
3. $\exists\, \beta > 0$ such that $\displaystyle\sup_{0 \neq \tau \in H} \frac{b(\tau, v)}{\|\tau\|_H} \geqslant \beta \|v\|_Q \qquad \forall\, v \in Q.$

[1] Theorem 3.4. C.I. Correa and G.N. Gatica, *On the continuous and discrete well-posedness of perturbed saddle-point formulations in Banach spaces*. Comput. Math. Appl. 117 (2022), 14–23.

Then, for each $(f, g) \in H' \times Q'$, there exists a unique $(\sigma, u) \in H \times Q$ solution to

$$\begin{aligned} a(\sigma,\tau) + b(\tau,u) &= f(\tau) \quad \forall\, \tau \in H, \\ b(\sigma,v) - c(u,v) &= g(v) \quad \forall\, v \in Q. \end{aligned}$$

Moreover, there exists a positive constant C, depending only on $\|a\|$, $\|c\|$, α, and β, such that

$$\|(\sigma,u)\|_{H\times Q} \leqslant C\left(\|f\|_{H'} + \|g\|_{Q'}\right).$$

Theorem: Well-posedness of BF (well-definedness of S)

Let $\delta > 0$. Given $(\mathbf{z}, \varphi) \in \mathcal{Q} \times \mathrm{H}^1(\Omega)$ such that $\|\mathbf{z}\|_{0,\rho;\Omega} \leqslant \delta$, the uncoupled problem BF has a unique solution $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathcal{H} \times \mathcal{Q}$, and, consequently, $\mathbf{S}(\mathbf{z}, \varphi)$ is well-defined. Moreover, there exists a positive constant $C_{\mathbf{S}}$, depending on δ, ρ, ν_0, ν_1, $\mathtt{D}_1$, $\mathtt{F}_1$, β and $|\Omega|$, such that

$$\|\mathbf{S}(\mathbf{z}, \varphi)\|_{0,\rho;\Omega} \leqslant \|(\boldsymbol{\sigma}, \mathbf{u})\|_{\mathcal{H}\times\mathcal{Q}}$$
$$\leqslant C_{\mathbf{S}} \Big\{ \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Omega} \big(\|\varphi\|_{1,\Omega} + \|\phi_{\mathrm{r}}\|_{0,s;\Omega} \big) \Big\}.$$

Theorem: Well-posedness of CDR (Well-definedness of $\widetilde{\mathbf{S}}$)

Let $\delta \in \left(0, \frac{1}{2}\,\|i_s\|^{-1}\,\min\{\kappa, \eta\}\right]$. Given $\mathbf{z} \in \mathcal{H}$ such that $\|\mathbf{z}\|_{0,\rho;\Omega} \leqslant \delta$, the uncoupled problem CDR has a unique solution $(\phi, \lambda) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma)$, and, consequently, $\widetilde{\mathbf{S}}(\mathbf{z})$ is well-defined. Furthermore, there exists a positive constant $C_{\widetilde{\mathbf{S}}}$, depending only on κ, η and $|\Omega|$, such that

$$\|\widetilde{\mathbf{S}}(\mathbf{z})\|_{1,\Omega} \leqslant \|(\phi, \lambda)\|_{\mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma)} \leqslant C_{\widetilde{\mathbf{S}}}\left\{\|g\|_{0,\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma}\right\}.$$

Why the restriction on δ?

Why the restriction on δ?

If $\delta \in \left(0, \frac{1}{2}\, \|i_s\|^{-1} \min\{\kappa, \eta\}\right]$,

$$\begin{aligned} a_{\mathbf{z}}(\psi, \psi) &= \kappa\, \|\nabla\psi\|_{0,\Omega}^2 + \int_\Omega (\mathbf{z} \cdot \nabla\psi)\, \psi + \eta\, \|\psi\|_{0,\Omega}^2 \\ &\geqslant \min\{\kappa, \eta\}\, \|\psi\|_{1,\Omega}^2 - \|i_s\|\, \|\mathbf{z}\|_{0,\rho;\Omega}\, \|\psi\|_{1,\Omega}^2 \\ &\geqslant \frac{1}{2} \min\{\kappa, \eta\}\, \|\psi\|_{1,\Omega}^2 \qquad \forall\, \psi \in \mathrm{H}^1(\Omega). \end{aligned}$$

We have established the well-definedness of $\mathbf{S}$ and $\widetilde{\mathbf{S}}$. Consequently, $\mathbf{T}$ is well-defined.

Next goal: Fixed-point equation

Find $\mathbf{u}$ such that

$$\mathbf{T}(\mathbf{u}) = \mathbf{u}.$$

For each $r \in (0, \delta]$, where $\delta := \frac{1}{2} \|i_s\|^{-1} \min\{\kappa, \eta\}$, define the closed and convex subset of $\mathcal{Q}$

$$\mathbf{W}(r) := \Big\{ \mathbf{z} \in \mathcal{Q} : \quad \|\mathbf{z}\|_{0,\rho;\,\Omega} \leqslant r \Big\}.$$

For each $r \in (0, \delta]$, where $\delta := \frac{1}{2} \|i_s\|^{-1} \min\{\kappa, \eta\}$, define the closed and convex subset of $\mathcal{Q}$

$$\mathbf{W}(r) := \Big\{ \mathbf{z} \in \mathcal{Q} : \quad \|\mathbf{z}\|_{0,\rho;\Omega} \leqslant r \Big\}.$$

1. $\mathbf{T}$ maps $\mathbf{W}(r)$ into itself (under small data assumption).
2. $\mathbf{S}$ is Lipschitz in the ball.
3. $\widetilde{\mathbf{S}}$ is Lipschitz in the ball.
4. $\mathbf{T}$ is Lipschitz in the ball.
5. If the data is small enough $\implies$ $\mathbf{T}$ is a contraction.

Define

$$C_{\mathtt{data}} = \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Omega} \big(\|\phi_{\mathrm{r}}\|_{0,s;\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} + \|g\|_{0,\Omega} \big).$$

Well-posedness of the continuous problem

Theorem

Let $r \in \big(0, \delta\big]$ and assume that the data satisfy

$$C_{\mathbf{T}}\, C_{\mathtt{data}} \leqslant r \qquad (\mathbf{T} \text{ maps ball into itself}),$$
$$L_{\mathbf{T}}\, C_{\mathtt{data}} < 1 \qquad (\mathbf{T} \text{ is a contraction}).$$

Then, there exists a unique $\mathbf{u} \in \mathbf{W}(r)$ such that $\mathbf{T}(\mathbf{u}) = \mathbf{u}$. Equivalently, our continuous problem has a unique solution $(\boldsymbol{\sigma}, \mathbf{u}, \phi, \lambda) \in \mathcal{H} \times \mathcal{Q} \times \mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma)$, with $\mathbf{u} \in \mathbf{W}(r)$. Moreover, there exist positive constants $\mathcal{C}_1$ and $\mathcal{C}_2$, depending on ρ, ν_0, ν_1, $\mathtt{D}_1$, $\mathtt{F}_1$, κ, η, β and $|\Omega|$, such that

$$\|(\boldsymbol{\sigma}, \mathbf{u})\|_{\mathcal{H}\times\mathcal{Q}} \leqslant \mathcal{C}_1\, C_{\mathtt{data}} \quad \text{and}$$
$$\|(\phi, \lambda)\|_{\mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma)} \leqslant \mathcal{C}_2 \left\{ \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} + \|g\|_{0,\Omega} \right\}.$$

Consider a regular family of triangulations $\{\mathcal{T}_h\}_{h>0}$ of $\bar{\Omega}$ made up of triangles K (when $n = 2$) or tetrahedra K (when $n = 3$) of diameter h_K, and set $h := \max\{h_K : \; K \in \mathcal{T}_h\}$.

Continuous	Discrete
$\mathbb{H}(\mathbf{div}_\ell;\Omega)$	$\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$
$\mathbf{L}^\rho(\Omega)$	$\mathbf{H}_h^{\mathbf{u}}$
$\mathrm{H}^1(\Omega)$	H_h^{ϕ}
$\mathrm{H}^{-1/2}(\Gamma)$	H_h^{λ}

Define $\mathbb{H}_h^{\boldsymbol{\sigma}} := \widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} \cap \mathbb{H}_0(\mathbf{div}_\ell;\Omega)$.

The Galerkin scheme

Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$ and $(\phi_h, \lambda_h) \in \mathrm{H}_h^{\phi} \times \mathrm{H}_h^{\lambda}$ such that

$$
\begin{array}{rcll}
\mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{u}_h) & = & \mathbf{F}(\boldsymbol{\tau}_h) & \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}\,, \\
\mathbf{b}(\boldsymbol{\sigma}_h, \mathbf{v}_h) - \mathbf{c}_{\mathbf{u}_h}(\mathbf{u}_h, \mathbf{v}_h) & = & \mathbf{G}_{\phi_h}(\mathbf{v}_h) & \forall\, \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}\,, \\
a_{\mathbf{u}_h}(\phi_h, \psi_h) + b(\psi_h, \lambda_h) & = & \mathrm{F}(\psi_h) & \forall\, \psi_h \in \mathrm{H}_h^{\phi}\,, \\
b(\phi_h, \xi_h) & = & \mathrm{G}(\xi_h) & \forall\, \xi_h \in \mathrm{H}_h^{\lambda}\,.
\end{array}
$$

Discrete Fixed-point strategy

Discrete uncoupled BF

$\mathbf{S}_{\mathtt{d}} : \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^{\phi} \rightarrow \mathbf{H}_h^{\mathbf{u}}$ such that $\mathbf{S}_{\mathtt{d}}(\mathbf{z}_h, \varphi_h) := \mathbf{u}_h$, where $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$ is the unique solution to

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{u}_h) &= \mathbf{F}(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}, \\ \mathbf{b}(\boldsymbol{\sigma}_h, \mathbf{v}_h) - \mathbf{c}_{\mathbf{z}_h}(\mathbf{u}_h, \mathbf{v}_h) &= \mathbf{G}_{\varphi_h}(\mathbf{v}_h) && \forall\, \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}. \end{aligned}$$

Discrete uncoupled CDR

$\widetilde{\mathbf{S}}_{\mathtt{d}} : \mathbf{H}_h^{\mathbf{u}} \rightarrow \mathrm{H}_h^{\phi}$ by $\widetilde{\mathbf{S}}_{\mathtt{d}}(\mathbf{z}_h) := \phi_h$, where $(\phi_h, \lambda_h) \in \mathrm{H}_h^{\phi} \times \mathrm{H}_h^{\lambda}$ is the unique solution to

$$\begin{aligned} a_{\mathbf{z}_h}(\phi_h, \psi_h) + b(\psi_h, \lambda_h) &= \mathrm{F}(\psi_h) && \forall\, \psi_h \in \mathrm{H}_h^{\phi}, \\ b(\phi_h, \xi_h) &= \mathrm{G}(\xi_h) && \forall\, \xi_h \in \mathrm{H}_h^{\lambda}, \end{aligned}$$

Discrete global fixed-point operator

$\mathbf{T}_{\mathtt{d}} : \mathbf{H}_h^{\mathbf{u}} \to \mathbf{H}_h^{\mathbf{u}}$ defined by

$$\mathbf{T}_{\mathtt{d}}(\mathbf{z}_h) := \mathbf{S}_{\mathtt{d}}\big(\mathbf{z}_h, \widetilde{\mathbf{S}}_{\mathtt{d}}(\mathbf{z}_h)\big) \qquad \forall\, \mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}.$$

Well-posedness of BF_h

Hypotheses on FE spaces

(H.0) $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$ contains the multiples of the identity tensor $\mathbb{I}$.

(H.1) $\mathbf{div}(\mathbb{H}_h^{\boldsymbol{\sigma}}) \subset \mathbf{H}_h^{\mathbf{u}}$.

$$\textbf{(H.0)} \implies \mathbb{H}_h^{\boldsymbol{\sigma}} = \left\{ \boldsymbol{\tau}_h - \left(\frac{1}{n\,|\Omega|} \int_\Omega \operatorname{tr}(\boldsymbol{\tau}_h) \right) \mathbb{I} : \quad \boldsymbol{\tau}_h \in \widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} \right\}.$$

Denote $\mathbb{V}_h$ as the kernel of $\mathbb{H}_h^{\boldsymbol{\sigma}} \ni \boldsymbol{\tau}_h \mapsto \mathbf{b}(\boldsymbol{\tau}_h, \cdot) \in (\mathbf{H}_h^{\mathbf{u}})'$.

(H.1) $\implies \mathbb{V}_h \subset \mathbb{V} \implies$ discrete inf-sup for $\mathbf{a}$ holds.

It remains to *prove* the discrete inf-sup of $\mathbf{b}$.

It remains to *prove* the discrete inf-sup of $\mathbf{b}$.

Another hypothesis

(H.2) There exists a positive constant $\beta_{\mathtt{d}}$, independent of h, such that

$$\sup_{0\neq\boldsymbol{\tau}_h\in\mathbb{H}_h^{\boldsymbol{\sigma}}} \frac{\mathbf{b}(\boldsymbol{\tau}_h,\mathbf{v}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_\ell;\Omega}} \geqslant \beta_{\mathtt{d}}\,\|\mathbf{v}_h\|_{0,\rho;\Omega} \qquad \forall\,\mathbf{v}_h\in\mathbf{H}_h^{\mathbf{u}}.$$

Well-posedness of BF_h

Theorem

Let $\delta_{\mathtt{d}} > 0$. Assume that **(H.0)**, **(H.1)** and **(H.2)** hold. Given $(\mathbf{z}_h, \varphi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^{\phi}$ such that $\|\mathbf{z}_h\|_{0,\rho;\Omega} \leqslant \delta_{\mathtt{d}}$, BF_h has a unique solution $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$. Consequently, $\mathbf{S}_{\mathtt{d}}(\mathbf{z}_h, \varphi_h)$ is well-defined, and there exists a positive constant $C_{\mathbf{S}_{\mathtt{d}}}$, depending only on $\delta_{\mathtt{d}}$, ρ, ν_0, ν_1, $\mathtt{D}_1$, $\mathtt{F}_1$, $\beta_{\mathtt{d}}$ and $|\Omega|$, such that

$$
\begin{aligned}
&\|\mathbf{S}_{\mathtt{d}}(\mathbf{z}_h, \psi_h)\|_{0,\rho;\Omega} \leqslant \|(\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathcal{H}\times\mathcal{Q}} \\
&\leqslant C_{\mathbf{S}_{\mathtt{d}}} \Big\{ \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Omega} \left(\|\varphi_h\|_{1,\Omega} + \|\phi_{\mathtt{r}}\|_{0,s;\Omega} \right) \Big\}.
\end{aligned}
$$

Well-posedness of CDR_h

Hypothesis

(H.3) There exists a positive constant $\widetilde{\beta}_{\mathtt{d}}$, independent of h, such that

$$\sup_{0 \neq \psi_h \in \mathrm{H}_h^{\phi}} \frac{b(\psi_h, \xi_h)}{\|\psi_h\|_{1,\Omega}} \geqslant \widetilde{\beta}_{\mathtt{d}} \, \|\xi_h\|_{-1/2,\Gamma} \qquad \forall\, \xi_h \in \mathrm{H}_h^{\lambda}.$$

Well-posedness of CDR_h

Theorem

Let $\delta_{\mathtt{d}} \in \left(0, \frac{1}{2}\|i_s\|^{-1} \min\{\kappa, \eta\}\right]$ and assume that **(H.3)** holds. Given $\mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}$ such that $\|\mathbf{z}_h\|_{0,\rho;\Omega} \leqslant \delta_{\mathtt{d}}$, CDR_h has a unique solution $(\phi_h, \lambda_h) \in \mathrm{H}_h^{\phi} \times \mathrm{H}_h^{\lambda}$, and, consequently, $\widetilde{\mathbf{S}}_{\mathtt{d}}(\phi_h, \lambda_h)$ is well-defined. Furthermore, there exists a positive constant $C_{\widetilde{\mathbf{S}}_{\mathtt{d}}}$, depending only on κ, η, $\widetilde{\beta}_{\mathtt{d}}$ and $|\Omega|$, such that

$$\|\widetilde{\mathbf{S}}_{\mathtt{d}}(\mathbf{z}_h)\|_{\mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma)} \leqslant C_{\widetilde{\mathbf{S}}_{\mathtt{d}}} \left\{\|g\|_{0,\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma}\right\}.$$

As in the continuous case, our main goal is to prove that there exists a unique solution to $\mathbf{T}_{\mathtt{d}}(\mathbf{u}_h) = \mathbf{u}_h$.

Let us introduce the discrete ball, depending on a parameter $r \in \big(0, \delta_{\mathtt{d}}\big]$, where $\delta_{\mathtt{d}} := \frac{1}{2}\,\|i_s\|^{-1} \min\{\kappa, \eta\}$, as

$$\mathbf{W}_h(r) := \Big\{ \mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}} : \quad \|\mathbf{z}_h\|_{0,\rho;\Omega} \leqslant r \Big\}.$$

1. $\mathbf{T}_{\mathtt{d}}$ maps $\mathbf{W}_h(r)$ into itself (under small data assumption).
2. $\mathbf{S}_{\mathtt{d}}$ is Lipschitz in the ball.
3. $\widetilde{\mathbf{S}}_{\mathtt{d}}$ is Lipschitz in the ball.
4. $\mathbf{T}_{\mathtt{d}}$ is Lipschitz in the ball.
5. If the data is small enough $\implies$ $\mathbf{T}_{\mathtt{d}}$ is a contraction.

Recall that

$$C_{\mathtt{data}} = \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Omega}\big(\|\phi_{\mathtt{r}}\|_{0,s;\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} + \|g\|_{0,\Omega}\big).$$

Well-posedness of the Galerkin scheme

Theorem

Let $r \in \big(0, \delta_{\mathtt{d}}\big]$ and assume that hypotheses **(H.0)** through **(H.3)** are satisfied. Furthermore, suppose that the data satisfy

$$C_{\mathbf{T}_{\mathtt{d}}}\, C_{\mathtt{data}} \leqslant r \quad \text{and} \quad L_{\mathbf{T}_{\mathtt{d}}}\, C_{\mathtt{data}} < 1.$$

Then, $\exists!\, \mathbf{u}_h \in \mathbf{W}_h(r) : \mathbf{T}_{\mathtt{d}}(\mathbf{u}_h) = \mathbf{u}_h$. Equivalently, the Galerkin scheme has a unique solution $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \phi_h, \lambda_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^{\phi} \times \mathrm{H}_h^{\lambda}$, with $\mathbf{u}_h \in \mathbf{W}_h(r)$. Moreover, there exist positive constants $\mathcal{C}_{1,\mathtt{d}}$ and $\mathcal{C}_{2,\mathtt{d}}$, depending only on ρ, ν_0, ν_1, $\mathtt{D}_1$, $\mathtt{F}_1$, κ, η, $\beta_{\mathtt{d}}$ and $|\Omega|$, such that

$$\big\|(\boldsymbol{\sigma}_h, \mathbf{u}_h)\big\|_{\mathcal{H}\times\mathcal{Q}} \leqslant \mathcal{C}_{1,\mathtt{d}}\, C_{\mathtt{data}} \quad \text{and}$$

$$\big\|(\phi_h, \lambda_h)\big\|_{\mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma)} \leqslant \mathcal{C}_{2,\mathtt{d}} \Big\{ \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} + \|g\|_{0,\Omega} \Big\}.$$

A priori error analysis

Under the previous assumptions and supposing that

$$\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} + \|g\|_{0,\Omega} + \|\phi_{\mathtt{r}}\|_{0,s;\,\Omega}$$

is small enough, there exists a positive constant $\mathcal{C}_{\mathtt{ST,MP}}$, independent of h, such that

$$\begin{aligned}
&\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathcal{H}\times\mathcal{Q}} + \|(\phi,\lambda) - (\phi_h,\lambda_h)\|_{\mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma)} \\
&\qquad \leqslant \mathcal{C}_{\mathtt{ST,MP}} \left\{ \operatorname{dist}\left((\boldsymbol{\sigma},\mathbf{u}), \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}\right) + \operatorname{dist}\left((\phi,\lambda), \mathrm{H}_h^{\phi} \times \mathrm{H}_h^{\lambda}\right) \right\}
\end{aligned}$$

Specific finite element subspaces

Notation

Given an integer $k \geqslant 0$, $S \subset \mathrm{R}^n$,

1. $\mathrm{P}_k(S)$ the space of polynomials of total degree at most k defined on S.
2. $\widetilde{\mathrm{P}}_k(K)$ is the space of polynomials of total degree equal to k defined on K
3. For each $K \in \mathcal{T}_h$, $\mathbf{RT}_k(K) := \mathbf{P}_k(K) \oplus \widetilde{\mathrm{P}}_k(K)\,\mathbf{x}$.

Let $\{\Gamma_1, \Gamma_2, \dots, \Gamma_m\}$ be an independent triangulation of Γ (made of straight segments in R^2, or triangles in R^3) and define

$$\widetilde{h} := \max_{j \in \{1,\dots,m\}} |\Gamma_j|.$$

$$
\begin{aligned}
\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} &:= \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div}_\ell;\Omega) : \quad \boldsymbol{\tau}_h\big|_K \in \mathbb{RT}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\},\\
\mathbf{H}_h^{\mathbf{u}} &:= \Big\{ \mathbf{v}_h \in \mathbf{L}^\rho(\Omega) : \quad \mathbf{v}_h\big|_K \in \mathbf{P}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\},\\
\mathrm{H}_h^{\phi} &:= \Big\{ \psi_h \in C(\overline{\Omega}) : \quad \psi_h|_K \in \mathrm{P}_{k+1}(K) \quad \forall\, K \in \mathcal{T}_h \Big\},\\
\mathrm{H}_{\tilde{h}}^{\lambda} &:= \Big\{ \xi_{\tilde{h}} \in \mathrm{L}^2(\Gamma) : \quad \xi_{\tilde{h}}|_{\Gamma_j} \in \mathrm{P}_k(\Gamma_j) \quad \forall\, j \in \{1,\dots,m\} \Big\}.
\end{aligned}
$$

1. **(H.0)** and **(H.1)** hold.
2. **(H.2)** holds[2].
3. There exists a positive constant C_0 such that for all $h \leqslant C_0 \, \widetilde{h}$ the discrete inf-sup for b holds, i.e. **(H.3)** holds[3].

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[2] Lemma 3.3. J. Camaño, C. Muñoz, and R. Oyarzúa, *Numerical analysis of a dual-mixed problem in non-standard Banach spaces*. Electron. Trans. Numer. Anal. 48 (2018), 114–130.

[3] Lemma 4.10. E. Colmenares, G.N. Gatica, and R. Oyarzúa, *Analysis of an augmented mixed-primal formulation for the stationary Boussinesq problem*. Numer. Methods Partial Differ. Equ. 32(2), 445-478 (2016)

Rates of convergence

Theorem

Suppose there exists $l \in (0, k+1]$ such that $\boldsymbol{\sigma} \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div}_\ell; \Omega)$, $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{l,\ell}(\Omega)$, $\mathbf{u} \in \mathbf{W}^{l,\rho}(\Omega)$, $\phi \in \mathrm{H}^{l+1}(\Omega)$ and $\lambda \in \mathrm{H}^{-1/2+l}(\Gamma)$. Then, for all $h \leqslant C_0\,\widetilde{h}$, there exists a positive constant C, independent of h and $\widetilde{h}$, such that

$$\begin{aligned}
&\|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathcal{H}\times\mathcal{Q}} + \|(\phi, \lambda) - (\phi_h, \lambda_h)\|_{\mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma)} \\
&\leqslant C\, h^l \left\{ \|\boldsymbol{\sigma}\|_{l,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{l,\ell;\Omega} + \|\mathbf{u}\|_{l,\rho;\Omega} + \|\phi\|_{l+1,\Omega} \right\} \\
&\quad + C\,\widetilde{h}^l\, \|\lambda\|_{-1/2+l,\Gamma}.
\end{aligned}$$

The fully-mixed approach

Recall our model problem:

$$\begin{aligned}
-\,\mathbf{div}\big(\nu\,\nabla\mathbf{u}\big) + \mathtt{D}\,\mathbf{u} + \mathtt{F}\,|\mathbf{u}|^{\rho-2}\,\mathbf{u} + \nabla p &= \mathbf{f}(\phi) && \text{in} \quad \Omega\,,\\
\mathrm{div}(\mathbf{u}) &= f && \text{in} \quad \Omega\,,\\
-\kappa\,\Delta\,\phi + \mathbf{u}\cdot\nabla\phi + \eta\,\phi &= g && \text{in} \quad \Omega\,,\\
\mathbf{u} = \mathbf{u}_{\mathrm{D}} \quad \text{and} \quad \phi &= \phi_{\mathrm{D}} && \text{on} \quad \Gamma\,,
\end{aligned}$$

Introduce: Pseudostress tensor $\boldsymbol{\sigma}$ and

$$\boxed{\text{Pseudodiffusion vector: } \boldsymbol{\vartheta} := \kappa \nabla\phi - \phi\,\mathbf{u} \quad \text{in} \quad \Omega\,.}$$

Taking divergence to the pseudodiffusion vector and using CDR equation:

$$\operatorname{div}(\boldsymbol{\vartheta}) = \underbrace{\kappa\,\Delta\phi - \mathbf{u}\cdot\nabla\phi}_{=\eta\,\phi - g} - f\,\phi = (\eta - f)\,\phi - g$$

$$\Longrightarrow \operatorname{div}(\boldsymbol{\vartheta}) - (\eta - f)\,\phi = -g \quad \text{in} \quad \Omega.$$

Find $\mathbf{u}$, $\boldsymbol{\sigma}$ and ϕ such that:

$$\begin{aligned}
-\mathbf{div}\big(\boldsymbol{\sigma}\big) + \mathtt{D}\,\mathbf{u} + \mathtt{F}\,|\mathbf{u}|^{\rho-2}\,\mathbf{u} &= \mathbf{f}(\phi) && \text{in} \quad \Omega \\
\frac{1}{\nu}\,\boldsymbol{\sigma}^{\mathrm{d}} &= \nabla\mathbf{u} - \frac{1}{n}\,f\,\mathbb{I} && \text{in} \quad \Omega \\
\mathrm{div}\big(\boldsymbol{\vartheta}\big) - (\eta - f)\,\phi &= -\,g && \text{in} \quad \Omega\,, \\
\kappa^{-1}\,\boldsymbol{\vartheta} &= \nabla\phi - \kappa^{-1}\,\phi\,\mathbf{u} && \text{in} \quad \Omega\,, \\
\mathbf{u} = \mathbf{u}_{\mathrm{D}}, \quad \phi &= \phi_{\mathrm{D}} && \text{on} \quad \Gamma \\
\int_\Omega \Big\{\mathrm{tr}(\boldsymbol{\sigma}) - \nu\,f\Big\} &= 0.
\end{aligned}$$

Integration by parts

For $t \in \begin{cases} (1, +\infty] \text{ in } \mathrm{R}^2\,, \\ [\frac{6}{5}, +\infty] \text{ in } \mathrm{R}^3\,, \end{cases}$ there holds

$$\langle \boldsymbol{\xi} \cdot \mathbf{n}, \varphi \rangle = \int_\Omega \Big\{ \boldsymbol{\xi} \cdot \nabla \varphi + \varphi \, \mathrm{div}(\boldsymbol{\xi}) \Big\} \quad \forall\, (\boldsymbol{\xi}, \varphi) \in \mathbf{H}(\mathrm{div}_t; \Omega) \times \mathrm{H}^1(\Omega)$$

and

$$\langle \boldsymbol{\tau} \mathbf{n}, \mathbf{v} \rangle = \int_\Omega \Big\{ \boldsymbol{\tau} : \nabla \mathbf{v} + \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\tau}) \Big\} \quad \forall\, (\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}(\mathbf{div}_t; \Omega) \times \mathbf{H}^1(\Omega)\,.$$

$$
\begin{aligned}
\operatorname{div}\big(\boldsymbol{\vartheta}\big) \,-\, (\eta \,-\, f)\,\phi &= -\,g && \text{in} \quad \Omega\,, \\
\kappa^{-1}\,\boldsymbol{\vartheta} &= \nabla\phi \,-\, \kappa^{-1}\,\phi\,\mathbf{u} && \text{in} \quad \Omega.
\end{aligned}
$$

Test the constitutive equation of CDR against $\boldsymbol{\psi} \in \mathbf{H}(\mathrm{div}_t;\,\Omega)$ and integrate by parts:

$$
\int_\Omega \kappa^{-1}\,\boldsymbol{\vartheta}\cdot\boldsymbol{\psi} + \int_\Omega \phi\,\mathrm{div}(\boldsymbol{\psi}) + \int_\Omega \kappa^{-1}\phi\,\mathbf{u}\cdot\boldsymbol{\psi} = \langle \boldsymbol{\psi}\cdot\mathbf{n}, \phi_{\mathrm{D}}\rangle
$$
$$
\forall\,\boldsymbol{\psi} \in \mathbf{H}(\mathrm{div}_t;\,\Omega),
$$

Test the momentum equation of CDR against a scalar field ξ:

$$
\int_\Omega \xi\,\mathrm{div}(\boldsymbol{\vartheta}) - \int_\Omega (\eta - f)\,\phi\,\xi = -\int_\Omega g\,\xi.
$$

$$
\begin{aligned}
\operatorname{div}(\boldsymbol{\vartheta}) - (\eta - f)\,\phi &= -g && \text{in} \quad \Omega\,,\\
\kappa^{-1}\,\boldsymbol{\vartheta} &= \nabla\phi - \kappa^{-1}\,\phi\,\mathbf{u} && \text{in} \quad \Omega.
\end{aligned}
$$

Test the constitutive equation of CDR against $\boldsymbol{\psi} \in \mathbf{H}(\operatorname{div}_t;\, \Omega)$ and integrate by parts:

$$
\int_\Omega \kappa^{-1}\,\boldsymbol{\vartheta}\cdot\boldsymbol{\psi} + \int_\Omega \phi\,\operatorname{div}(\boldsymbol{\psi}) + \int_\Omega \kappa^{-1}\phi\,\mathbf{u}\cdot\boldsymbol{\psi} = \langle \boldsymbol{\psi}\cdot\mathbf{n}, \phi_{\mathrm{D}}\rangle
$$

$$
\forall\,\boldsymbol{\psi} \in \mathbf{H}(\operatorname{div}_t;\, \Omega),
$$

Test the momentum equation of CDR against a scalar field ξ:

$$
\int_\Omega \xi\,\operatorname{div}(\boldsymbol{\vartheta}) - \int_\Omega (\eta - f)\,\phi\,\xi = -\int_\Omega g\,\xi.
$$

$\boxed{\text{Now } \nabla\phi \text{ was eliminated!!!}}$

$$\begin{aligned} \mathrm{div}\big(\boldsymbol{\vartheta}\big) - (\eta - f)\,\phi &= -g && \text{in} \quad \Omega\,, \\ \kappa^{-1}\,\boldsymbol{\vartheta} &= \nabla\phi - \kappa^{-1}\,\phi\,\mathbf{u} && \text{in} \quad \Omega. \end{aligned}$$

Test the constitutive equation of CDR against $\boldsymbol{\psi} \in \mathbf{H}(\mathrm{div}_t;\,\Omega)$ and integrate by parts:

$$\int_\Omega \kappa^{-1}\,\boldsymbol{\vartheta}\cdot\boldsymbol{\psi} + \int_\Omega \phi\,\mathrm{div}(\boldsymbol{\psi}) + \int_\Omega \kappa^{-1}\phi\,\mathbf{u}\cdot\boldsymbol{\psi} = \langle \boldsymbol{\psi}\cdot\mathbf{n}, \phi_{\mathrm{D}}\rangle$$

$$\forall\,\boldsymbol{\psi} \in \mathbf{H}(\mathrm{div}_t;\,\Omega),$$

Test the momentum equation of CDR against a scalar field ξ:

$$\int_\Omega \xi\,\mathrm{div}(\boldsymbol{\vartheta}) - \int_\Omega (\eta - f)\,\phi\,\xi = -\int_\Omega g\,\xi.$$

Now $\nabla\phi$ was eliminated!!! $\Longrightarrow$ $\phi \in \mathrm{L}^s(\Omega)$ $\Longrightarrow$ t conjugate of s

$$\begin{aligned} \operatorname{div}\big(\boldsymbol{\vartheta}\big) - (\eta - f)\,\phi &= -g && \text{in} \quad \Omega\,, \\ \kappa^{-1}\,\boldsymbol{\vartheta} &= \nabla\phi - \kappa^{-1}\,\phi\,\mathbf{u} && \text{in} \quad \Omega. \end{aligned}$$

Test the constitutive equation of CDR against $\boldsymbol{\psi} \in \mathbf{H}(\mathrm{div}_t;\,\Omega)$ and integrate by parts:

$$\int_\Omega \kappa^{-1}\,\boldsymbol{\vartheta}\cdot\boldsymbol{\psi} + \int_\Omega \phi\,\mathrm{div}(\boldsymbol{\psi}) + \int_\Omega \kappa^{-1}\phi\,\mathbf{u}\cdot\boldsymbol{\psi} = \langle \boldsymbol{\psi}\cdot\mathbf{n}, \phi_{\mathrm{D}}\rangle$$
$$\forall\,\boldsymbol{\psi} \in \mathbf{H}(\mathrm{div}_t;\,\Omega),$$

Test the momentum equation of CDR against a scalar field ξ:

$$\int_\Omega \xi\,\mathrm{div}(\boldsymbol{\vartheta}) - \int_\Omega (\eta - f)\,\phi\,\xi = -\int_\Omega g\,\xi.$$

$$\boxed{s = 2\rho/(\rho - 2) \quad \text{and} \quad t = 2\rho/(\rho + 2).}$$

Summary:

$$\ell := \frac{\rho}{\rho - 1} \in \left[\frac{4}{3}, \frac{3}{2}\right], \qquad s := \frac{2\rho}{\rho - 2} \in [4, 6]$$

$$\text{and} \quad t := \frac{2\rho}{\rho + 2} \in \left[\frac{6}{5}, \frac{4}{3}\right],$$

Define

$$\mathcal{H} := \mathbb{H}_0(\mathbf{div}_\ell; \Omega), \quad \mathcal{Q} := \mathbf{L}^\rho(\Omega),$$

$$\mathrm{X} := \mathbf{H}(\mathrm{div}_t; \Omega) \quad \text{and} \quad \mathrm{Y} := \mathrm{L}^s(\Omega)$$

BF coupled with CDR — Fully-mixed formulation

Find $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\vartheta}, \phi) \in \mathcal{H} \times \mathcal{Q} \times \mathrm{X} \times \mathrm{Y}$ such that

$$
\begin{array}{llllll}
\displaystyle\int_\Omega \frac{1}{\nu}\boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} & \displaystyle + \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) & & = & \displaystyle\langle \boldsymbol{\tau}\,\mathbf{n}, \mathbf{u}_{\mathrm{D}}\rangle - \frac{1}{n}\int_\Omega f\,\mathrm{tr}(\boldsymbol{\tau}), \\
\displaystyle\int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) & \displaystyle - \int_\Omega \mathtt{D}\,\mathbf{u}\cdot\mathbf{v} - \int_\Omega \mathtt{F}\,|\mathbf{u}|^{\rho-2}\mathbf{u}\cdot\mathbf{v} & & = & \displaystyle - \int_\Omega \mathbf{f}(\phi)\cdot\mathbf{v}, \\
\displaystyle\int_\Omega \kappa^{-1}\boldsymbol{\vartheta}\cdot\boldsymbol{\psi} & \displaystyle + \int_\Omega \phi\,\mathrm{div}(\boldsymbol{\psi}) & \underbrace{\displaystyle + \int_\Omega \kappa^{-1}\phi\,\mathbf{u}\cdot\boldsymbol{\psi}}_{\text{“Nonlinear”}} & = & \langle \boldsymbol{\psi}\cdot\mathbf{n}, \phi_{\mathrm{D}}\rangle \\
\displaystyle\int_\Omega \xi\,\mathrm{div}(\boldsymbol{\vartheta}) & \displaystyle - \int_\Omega (\eta - f)\,\phi\,\xi & & = & \displaystyle - \int_\Omega g\,\xi.
\end{array}
$$

$$\forall\,(\boldsymbol{\tau}, \mathbf{v}, \psi, \xi) \in \mathcal{H} \times \mathcal{Q} \times \mathrm{X} \times \mathrm{Y}.$$

Definitions

The bilinear forms $\widehat{a} : \mathrm{X} \times \mathrm{X} \to \mathrm{R}$, $\widehat{b} : \mathrm{X} \times \mathrm{Y} \to \mathrm{R}$, $d_{\mathbf{z}} : \mathrm{X} \times \mathrm{Y} \to \mathrm{R}$ for each $\mathbf{z} \in \mathcal{Q}$, $\widehat{c}_f : \mathrm{Y} \times \mathrm{Y} \to \mathrm{R}$, and the linear functionals $\widehat{\mathrm{F}} : \mathrm{X} \to \mathrm{R}$ and $\widehat{\mathrm{G}} : \mathrm{Y} \to \mathrm{R}$, are defined as

$$\begin{aligned}
\widehat{a}(\boldsymbol{\zeta}, \boldsymbol{\psi}) &:= \int_\Omega \kappa^{-1} \boldsymbol{\zeta} \cdot \boldsymbol{\psi}, & \widehat{b}(\boldsymbol{\psi}, \xi) &:= \int_\Omega \xi \, \mathrm{div}(\boldsymbol{\psi}), \\
d_{\mathbf{z}}(\boldsymbol{\psi}, \xi) &:= \int_\Omega \kappa^{-1} \xi \, \mathbf{z} \cdot \boldsymbol{\psi}, & \widehat{c}_f(\zeta, \xi) &:= \int_\Omega (\eta - f) \, \zeta \, \xi, \\
\widehat{\mathrm{F}}(\boldsymbol{\psi}) &:= \langle \boldsymbol{\psi} \cdot \mathbf{n}, \phi_{\mathrm{D}} \rangle, & \widehat{\mathrm{G}}(\xi) &:= -\int_\Omega g \, \xi.
\end{aligned}$$

BF coupled with CDR — Fully-mixed formulation

Find $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathcal{H} \times \mathcal{Q}$ and $(\boldsymbol{\vartheta}, \phi) \in \mathrm{X} \times \mathrm{Y}$ such that

$$\begin{aligned}
\mathbf{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau}, \mathbf{u}) &= \mathbf{F}(\boldsymbol{\tau}) && \forall\, \boldsymbol{\tau} \in \mathcal{H}\,, \\
\mathbf{b}(\boldsymbol{\sigma}, \mathbf{v}) - \mathbf{c}_{\mathbf{u}}(\mathbf{u}, \mathbf{v}) &= \mathbf{G}_{\phi}(\mathbf{v}) && \forall\, \mathbf{v} \in \mathcal{Q}\,, \\
\widehat{a}(\boldsymbol{\vartheta}, \boldsymbol{\psi}) + \widehat{b}(\boldsymbol{\psi}, \phi) + d_{\mathbf{u}}(\boldsymbol{\psi}, \phi) &= \widehat{\mathrm{F}}(\boldsymbol{\psi}) && \forall\, \boldsymbol{\psi} \in \mathrm{X}\,, \\
\widehat{b}(\boldsymbol{\vartheta}, \xi) - \widehat{c}_f(\phi, \xi) &= \widehat{\mathrm{G}}(\xi) && \forall\, \xi \in \mathrm{Y}\,,
\end{aligned}$$

A fixed-point strategy

Uncoupled mixed CDR

$\widehat{\mathbf{S}} : \mathcal{Q} \to \mathrm{Y}$ by $\widehat{\mathbf{S}}(\mathbf{z}) := \phi$, where $(\boldsymbol{\vartheta}, \phi) \in \mathrm{X} \times \mathrm{Y}$ is the unique solution to

$$\begin{array}{rcll} \widehat{a}(\boldsymbol{\vartheta}, \boldsymbol{\psi}) + \widehat{b}(\boldsymbol{\psi}, \phi) + d_{\mathbf{z}}(\boldsymbol{\psi}, \phi) & = & \widehat{\mathrm{F}}(\boldsymbol{\psi}) & \forall\, \boldsymbol{\psi} \in \mathrm{X}\,, \\ \widehat{b}(\boldsymbol{\vartheta}, \xi) - \widehat{c}_f(\phi, \xi) & = & \widehat{\mathrm{G}}(\xi) & \forall\, \xi \in \mathrm{Y}\,. \end{array}$$

Equivalently, $(\boldsymbol{\vartheta}, \phi) \in \mathrm{X} \times \mathrm{Y}$ is the unique solution to

$$\widehat{\mathbf{A}}\big((\boldsymbol{\vartheta}, \phi), (\boldsymbol{\psi}, \xi)\big) + d_{\mathbf{z}}(\boldsymbol{\psi}, \phi) = \widehat{\mathrm{F}}(\boldsymbol{\psi}) + \widehat{\mathrm{G}}(\xi) \qquad \forall\, (\boldsymbol{\psi}, \xi) \in \mathrm{X} \times \mathrm{Y},$$

where $\widehat{\mathbf{A}} : (\mathrm{X} \times \mathrm{Y}) \times (\mathrm{X} \times \mathrm{Y}) \to \mathrm{R}$ is the bilinear form defined by

$$\widehat{\mathbf{A}}\big((\boldsymbol{\varrho}, \zeta), (\boldsymbol{\psi}, \xi)\big) := \widehat{a}(\boldsymbol{\varrho}, \boldsymbol{\psi}) + \widehat{b}(\boldsymbol{\psi}, \zeta) + \widehat{b}(\boldsymbol{\varrho}, \xi) - \widehat{c}_f(\zeta, \xi).$$

Global fixed-point operator

$\widehat{\mathbf{T}} : \mathcal{Q} \to \mathcal{Q}$,

$$\widehat{\mathbf{T}}(\mathbf{z}) := \mathbf{S}\big(\mathbf{z}, \widehat{\mathbf{S}}(\mathbf{z})\big) \qquad \forall\, \mathbf{z} \in \mathcal{Q}.$$

Lemma

Suppose that

$$f(\mathbf{x}) \leqslant \eta \qquad \forall\, \mathbf{x} \in \Omega.$$

Then, there exists a positive constant $\alpha_{\widehat{\mathbf{A}}}$, depending on κ, η, ρ and $|\Omega|$, such that

$$\sup_{\mathbf{0} \neq (\boldsymbol{\psi}, \xi) \in \mathrm{X} \times \mathrm{Y}} \frac{\widehat{\mathbf{A}}((\boldsymbol{\varrho}, \zeta), (\boldsymbol{\psi}, \xi))}{\|(\boldsymbol{\psi}, \xi)\|_{\mathrm{X} \times \mathrm{Y}}} \geqslant \alpha_{\widehat{\mathbf{A}}} \|(\boldsymbol{\varrho}, \zeta)\|_{\mathrm{X} \times \mathrm{Y}}$$

$$\forall\, (\boldsymbol{\varrho}, \zeta) \in \mathrm{X} \times \mathrm{Y}.$$

Sketch of the proof

Prove the well-posedness of the associated problem using Correa & Gatica[4].

We need $\widehat{c}_f$ to be positive semi-definite:

$$\widehat{c}_f(\xi, \xi) = \int_\Omega \underbrace{(\eta - f)} \, |\xi|^2 \geqslant 0 \qquad \forall\, \xi \in \mathrm{Y}.$$

Use the assumption!

[4] Theorem 3.4. C.I. Correa and G.N. Gatica, *On the continuous and discrete well-posedness of perturbed saddle-point formulations in Banach spaces*. Comput. Math. Appl. 117 (2022), 14–23.

Recall that $d_{\mathbf{z}}(\boldsymbol{\psi},\zeta) := \int_\Omega \kappa^{-1}\,\zeta\,\mathbf{z}\cdot\boldsymbol{\psi}$. One has

$$\left|d_{\mathbf{z}}(\boldsymbol{\psi},\zeta)\right| \leqslant \kappa^{-1}\,\|\mathbf{z}\|_{0,\rho;\Omega}\,\|\boldsymbol{\psi}\|_{\mathrm{X}}\,\|\zeta\|_{\mathrm{Y}} \leqslant \frac{\alpha_{\widehat{\mathbf{A}}}}{2}\,\|\boldsymbol{\psi}\|_{\mathrm{X}}\,\|\zeta\|_{\mathrm{Y}},$$

provided that $\|\mathbf{z}\|_{0,\rho;\,\Omega} \leqslant \frac{1}{2}\,\kappa\,\alpha_{\widehat{\mathbf{A}}}$. Consequently,

$$\begin{aligned}
\sup_{\mathbf{0}\neq(\boldsymbol{\psi},\xi)\in\mathrm{X}\times\mathrm{Y}} & \frac{\widehat{\mathbf{A}}((\boldsymbol{\varrho},\zeta),(\boldsymbol{\psi},\xi)) + d_{\mathbf{z}}(\boldsymbol{\psi},\zeta)}{\|(\boldsymbol{\psi},\xi)\|_{\mathrm{X}\times\mathrm{Y}}} \\
& \geqslant \sup_{\mathbf{0}\neq(\boldsymbol{\psi},\xi)\in\mathrm{X}\times\mathrm{Y}} \frac{\widehat{\mathbf{A}}((\boldsymbol{\varrho},\zeta),(\boldsymbol{\psi},\xi))}{\|(\boldsymbol{\psi},\xi)\|_{\mathrm{X}\times\mathrm{Y}}} - \frac{\alpha_{\widehat{\mathbf{A}}}}{2}\,\|\zeta\|_{\mathrm{Y}} \\
& \geqslant \frac{\alpha_{\widehat{\mathbf{A}}}}{2}\,\|(\boldsymbol{\varrho},\zeta)\|_{\mathrm{X}\times\mathrm{Y}},
\end{aligned}$$

for all $(\boldsymbol{\varrho},\zeta)\in\mathrm{X}\times\mathrm{Y}$.

Well-posedness uncoupled mixed CDR

Theorem

Let $\delta \in \left(0, \frac{1}{2}\,\kappa\,\alpha_{\widehat{\mathbf{A}}}\right]$ and suppose that $f(\mathbf{x}) \leqslant \eta$ for all $\mathbf{x} \in \Omega$. Given $\mathbf{z} \in \mathcal{Q}$ such that $\|\mathbf{z}\|_{0,\rho;\,\Omega} \leqslant \delta$, CDR (mixed) has a unique solution $(\boldsymbol{\vartheta}, \phi) \in \mathrm{X} \times \mathrm{Y}$ and, consequently, $\widehat{\mathbf{S}}(\mathbf{z})$ is well-defined. Moreover, there exists a positive constant $C_{\widehat{\mathbf{S}}}$, depending on κ, η, ρ and $|\Omega|$, such that

$$\|\widehat{\mathbf{S}}(\mathbf{z})\|_{0,s;\Omega} \leqslant \|(\boldsymbol{\vartheta}, \phi)\|_{\mathrm{X}\times\mathrm{Y}} \leqslant C_{\widehat{\mathbf{S}}} \left\{ \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} + \|g\|_{0,t;\Omega} \right\}.$$

Well-posedness of the continuous fully-mixed problem

Given $r \in \big(0, \delta\big]$, with $\delta := \frac{1}{2}\,\kappa\,\alpha_{\widehat{\mathbf{A}}}$, we define $\mathbf{W}(r)$ as the closed and convex subset of $\mathcal{Q}$ given by

$$\mathbf{W}(r) := \Big\{ \mathbf{z} \in \mathcal{Q} : \quad \|\mathbf{z}\|_{0,\rho;\Omega} \leqslant r \Big\}.$$

Under assumptions of small data,

1. $\widehat{\mathbf{T}}$ maps $\mathbf{W}(r)$ into itself.
2. $\mathbf{S}$ is Lipschitz.
3. $\widehat{\mathbf{S}}$ is Lipschitz.
4. $\widehat{\mathbf{T}}$ is Lipschitz.

Then, if the data is small enough so that $\widehat{\mathbf{T}}$ is a contraction, we have the well-posedness of our fully-mixed formulation.

Discrete setting

Continuous	Discrete
$\mathbb{H}(\mathbf{div}_\ell;\Omega)$	$\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$
$\mathbf{L}^\rho(\Omega)$	$\mathbf{H}_h^{\mathbf{u}}$
$\mathbf{H}(\mathrm{div}_t;\Omega)$	$\mathbf{H}_h^{\boldsymbol{\vartheta}}$
$\mathrm{L}^s(\Omega)$	$\widehat{\mathrm{H}}_h^{\phi}$

Define $\mathbb{H}_h^{\boldsymbol{\sigma}} := \widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} \cap \mathbb{H}_0(\mathbf{div}_\ell;\Omega)$.

The Galerkin scheme

Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$ and $(\boldsymbol{\vartheta}_h, \phi_h) \in \mathbf{H}_h^{\boldsymbol{\vartheta}} \times \widehat{\mathrm{H}}_h^{\phi}$ such that

$$
\begin{array}{rcll}
\mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{u}_h) & = & \mathbf{F}(\boldsymbol{\tau}_h) & \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}\,, \\
\mathbf{b}(\boldsymbol{\sigma}_h, \mathbf{v}_h) - \mathbf{c}_{\mathbf{u}_h}(\mathbf{u}_h, \mathbf{v}_h) & = & \mathbf{G}_{\phi_h}(\mathbf{v}_h) & \forall\, \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}\,, \\
\widehat{a}(\boldsymbol{\vartheta}_h, \boldsymbol{\psi}_h) + \widehat{b}(\boldsymbol{\psi}_h, \phi_h) + d_{\mathbf{u}_h}(\boldsymbol{\psi}_h, \phi_h) & = & \widehat{\mathrm{F}}(\boldsymbol{\psi}_h) & \forall\, \boldsymbol{\psi}_h \in \mathbf{H}_h^{\boldsymbol{\vartheta}}\,, \\
\widehat{b}(\boldsymbol{\vartheta}_h, \xi_h) - \widehat{c}_f(\phi_h, \xi_h) & = & \widehat{\mathrm{G}}(\xi_h) & \forall\, \xi_h \in \widehat{\mathrm{H}}_h^{\phi}\,.
\end{array}
$$

Specific finite element subspaces

$$\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} := \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div}_\ell;\Omega) : \quad \boldsymbol{\tau}_h\big|_K \in \mathbb{RT}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\},$$

$$\mathbf{H}_h^{\mathbf{u}} := \Big\{ \mathbf{v}_h \in \mathbf{L}^\rho(\Omega) : \quad \mathbf{v}_h\big|_K \in \mathbf{P}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\},$$

$$\mathbf{H}_h^{\boldsymbol{\vartheta}} := \Big\{ \boldsymbol{\psi}_h \in \mathbf{H}(\mathrm{div}_t;\Omega) : \quad \boldsymbol{\psi}_h\big|_K \in \mathbf{RT}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\},$$

$$\widehat{\mathrm{H}}_h^{\phi} := \Big\{ \xi_h \in \mathrm{L}^s(\Omega) : \quad \xi_h\big|_K \in \mathrm{P}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\},$$

Rates of convergence

Assume that there exists $l \in (0, k+1]$ such that $\boldsymbol{\sigma} \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div}_\ell; \Omega)$, $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{l,\ell}(\Omega)$, $\mathbf{u} \in \mathbf{W}^{l,\rho}(\Omega)$, $\boldsymbol{\vartheta} \in \mathbf{H}^l(\Omega)$, $\mathrm{div}(\boldsymbol{\vartheta}) \in \mathrm{W}^{l,t}(\Omega)$ and $\phi \in \mathrm{W}^{l,s}(\Omega)$. Then, there exists a positive constant C, independent of h, such that

$$\begin{aligned}
&\|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathcal{H}\times\mathcal{Q}} + \|(\boldsymbol{\vartheta}, \phi) - (\boldsymbol{\vartheta}_h, \phi_h)\|_{\mathrm{X}\times\mathrm{Y}} \\
&\leqslant C\, h^l \Big\{ \|\boldsymbol{\sigma}\|_{l,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{l,\ell;\Omega} + \|\mathbf{u}\|_{l,\rho;\Omega} \\
&\qquad + \|\boldsymbol{\vartheta}\|_{l,\Omega} + \|\mathrm{div}(\boldsymbol{\vartheta})\|_{l,t;\Omega} + \|\phi\|_{l,s;\Omega} \Big\}.
\end{aligned}$$

Numerical tests

Post-processing formula for the pressure:

$$p_h \;=\; -\frac{1}{n}\mathrm{tr}(\boldsymbol{\sigma}_h) + \frac{\nu}{n}\, f\,.$$

Setting

We set $\kappa = 1$, $\eta = 1$, and $\phi_{\mathtt{r}} = 0$, and choose the Brinkman, Darcy, and Forchheimer coefficients as follows:

$$\nu(\mathbf{x}) = \exp\left(-\prod_{i=1}^{n} x_i\right)\,, \quad \mathtt{D}(\mathbf{x}) = \exp\left(-\sum_{i=1}^{n} x_i\right)\,,$$

$$\text{and} \quad \mathtt{F}(\mathbf{x}) = \exp\left(\sum_{i=1}^{n} x_i\right)\,,$$

First example: 2D with manufactured solutions

$\Omega = (0,1)^2$, $\rho = 3$, so that $\ell = 3/2$, $s = 6$, and $t = 6/5$. Take $\mathbf{g} = (0,-1)^{\mathrm{t}}$ and adjust the data $\mathbf{f}(\phi)$, f, and g so that the exact solution is given by

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} \cos(\pi x_1)\,\sin(\pi x_2) \\ \sin(\pi x_1)\,\exp(x_2) \end{pmatrix}, \quad p(\mathbf{x}) = \cos(\pi x_1)\,\sin(\pi x_2)\,,$$

$$\text{and} \quad \phi(\mathbf{x}) = 0.1 + 0.3\,\exp(x_1 x_2)\,.$$

In this example, it is not true that

$$f(\mathbf{x}) \leqslant \eta \qquad \forall\, \mathbf{x} \in \Omega.$$

First example: 2D with manufactured solutions ($k = 0$)

Mixed-primal $\mathbb{RT}_k - \mathbf{P}_k - \mathrm{P}_{k+1} - \mathrm{P}_k$ scheme with $k = 0$											
DOF	h	it	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e(ϕ)	r(ϕ)	$\widetilde{h}$	e(λ)	r(λ)
914	0.196	4	1.6E+00	–	1.5E-01	–	3.3E-02	–	0.250	1.1E-01	–
2010	0.127	4	1.0E+00	0.966	1.0E-01	0.930	2.2E-02	0.896	0.167	7.0E-02	1.052
5434	0.078	4	6.2E-01	1.063	6.0E-02	1.082	1.3E-02	1.076	0.100	4.1E-02	1.050
17551	0.044	4	3.4E-01	1.064	3.3E-02	1.083	7.0E-03	1.098	0.056	2.2E-02	1.044
60936	0.024	4	1.8E-01	1.054	1.8E-02	1.044	3.7E-03	1.060	0.029	1.2E-02	1.023
227621	0.014	4	9.4E-02	1.108	9.1E-03	1.107	1.9E-03	1.096	0.015	5.9E-03	1.009

Fully-mixed $\mathbb{RT}_k - \mathbf{P}_k - \mathbf{RT}_k - \mathrm{P}_k$ scheme with $k = 0$										
DOF	h	it	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\boldsymbol{\vartheta}$)	r($\boldsymbol{\vartheta}$)	$\widehat{\mathrm{e}}(\phi)$	$\widehat{\mathrm{r}}(\phi)$
1188	0.196	4	1.6E+00	–	1.5E-01	–	1.7E-01	–	1.8E-02	–
2652	0.127	4	1.0E+00	0.966	1.0E-01	0.930	1.2E-01	0.914	1.4E-02	0.575
7260	0.078	4	6.2E-01	1.063	6.0E-02	1.082	6.7E-02	1.086	8.6E-03	1.003
23661	0.044	4	3.4E-01	1.064	3.3E-02	1.083	3.6E-02	1.104	4.6E-03	1.121
82578	0.024	4	1.8E-01	1.054	1.8E-02	1.044	2.0E-02	1.031	2.6E-03	0.948
309387	0.014	4	9.4E-02	1.108	9.1E-03	1.107	1.0E-02	1.118	1.3E-03	1.132

First example: 2D with manufactured solutions ($k = 1$)

Mixed-primal $\mathbb{RT}_k - \mathbf{P}_k - \mathrm{P}_{k+1} - \mathrm{P}_k$ scheme with $k = 1$											
DOF	h	it	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e(ϕ)	r(ϕ)	$\widetilde{h}$	e(λ)	r(λ)
2891	0.196	4	9.3E-02	–	7.2E-03	–	1.1E-03	–	0.250	4.8E-03	–
6427	0.127	4	4.0E-02	1.927	3.2E-03	1.883	4.8E-04	1.986	0.167	2.1E-03	2.013
17531	0.078	4	1.5E-02	2.060	1.2E-03	1.960	1.6E-04	2.210	0.100	7.6E-04	2.025
56983	0.044	4	4.4E-03	2.116	3.7E-04	2.106	4.5E-05	2.275	0.056	2.3E-04	2.006
198563	0.024	4	1.3E-03	2.111	1.0E-04	2.135	1.3E-05	2.158	0.029	6.5E-05	2.003
743263	0.014	4	3.4E-04	2.220	2.8E-05	2.238	3.3E-06	2.270	0.015	1.7E-05	2.002

Fully-mixed $\mathbb{RT}_k - \mathbf{P}_k - \mathbf{RT}_k - \mathrm{P}_k$ scheme with $k = 1$										
DOF	h	it	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\boldsymbol{\vartheta}$)	r($\boldsymbol{\vartheta}$)	$\widehat{\mathrm{e}}(\phi)$	$\widehat{\mathrm{r}}(\phi)$
3744	0.196	4	9.3E-02	–	7.2E-03	–	1.1E-02	–	5.3E-04	–
8400	0.127	4	4.0E-02	1.927	3.2E-03	1.883	4.8E-03	1.776	2.8E-04	1.446
23088	0.078	4	1.5E-02	2.060	1.2E-03	1.960	1.7E-03	2.082	1.1E-04	1.856
75456	0.044	4	4.4E-03	2.116	3.7E-04	2.106	5.3E-04	2.123	3.3E-05	2.227
263760	0.024	4	1.3E-03	2.111	1.0E-04	2.135	1.5E-04	2.097	1.1E-05	1.908
989088	0.014	4	3.4E-04	2.220	2.8E-05	2.238	4.1E-05	2.229	2.7E-06	2.274

Second example: 3D with manufactured solutions

$\Omega = (0,1)^3$. Choose $\rho = 7/2$, whence $\ell = 7/5$, $s = 14/3$, and $t = 14/11$. The solution is given by

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} \sin(\pi x_1)\,\sin(\pi x_2)\,\sin(\pi x_3) \\ -\cos(\pi x_1)\,\cos(\pi x_2)\,\cos(\pi x_3) \\ \cos(\pi x_1)\,\cos(\pi x_2)\,\sin(\pi x_3) \end{pmatrix},$$

$$p(\mathbf{x}) = \cos(\pi x_1)\,\exp(x_2 + x_3)\,,$$

$$\text{and} \quad \phi(\mathbf{x}) = 0.1 + 0.3\,\exp(x_1 x_2 x_3)\,.$$

Here, $\mathbf{g} = (0,0,-1)^{\mathrm{t}}$, while the data $\mathbf{f}(\phi)$, f, and g are computed using the solution above.

Second example: 3D with manufactured solutions ($k = 0$ Fully-mixed)

Fully-mixed $\mathbb{RT}_k - \mathbf{P}_k - \mathbf{RT}_k - \mathrm{P}_k$ scheme with $k = 0$										
DOF	h	it	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\boldsymbol{\vartheta}$)	r($\boldsymbol{\vartheta}$)	e(ϕ)	r(ϕ)
672	0.866	4	1.0E+01	–	3.8E-01	–	5.1E-01	–	6.6E-02	–
4992	0.433	4	5.6E+00	0.887	2.0E-01	0.916	2.7E-01	0.906	3.6E-02	0.880
38400	0.217	4	2.8E+00	0.999	1.0E-01	0.980	1.4E-01	0.974	1.8E-02	0.973
301056	0.108	4	1.4E+00	1.027	5.0E-02	1.001	6.9E-02	0.995	9.2E-03	0.994
585600	0.087	4	1.1E+00	1.022	4.0E-02	1.002	5.6E-02	0.998	7.4E-03	0.998

Second example: 3D with manufactured solutions ($k = 0$ Fully-mixed)

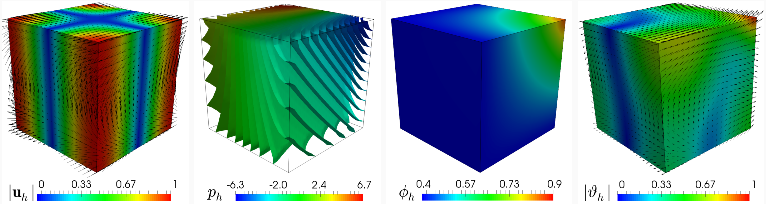

Figure 2: [Example 2] Computed magnitude of the velocity, pressure and concentration fields, and magnitude of the pseudodiffusion vector.

Example 3: Fluid flow through a rectangular domain with circular obstacles

- $\Omega = (0,2) \times (0,0.25) \backslash \Omega_c$, where Ω_c represents circular obstacles, with boundary $\Gamma = \partial\Omega$, where the input and output parts are defined as $\Gamma_{\mathrm{in}} = \{0\} \times (0,0.25)$ and $\Gamma_{\mathrm{out}} = \{2\} \times (0,0.25)$.
- $\rho = 4$, $\mathbf{g} = (0,-9.81)^{\mathrm{t}}$, $f = 0$, and $g = 0$.
- The initial conditions for both the velocity and concentration are taken to be zero.

Denoting $u_{\mathrm{in}} := -10\, x_2(x_2 - 0.25)\big(1 + 0.5\sin(2\pi\, t/T)\big)$ and $\phi_{\mathrm{in}} := 5 + 0.5\sin(2\pi\, t)$, the boundary conditions are given by

$$\mathbf{u} = (u_{\mathrm{in}}, 0)^{\mathrm{t}} \quad \text{on} \quad \Gamma_{\mathrm{in}}\,, \quad \mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma \backslash \big(\Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}}\big)\,, \quad \boldsymbol{\sigma}\mathbf{n} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{out}}\,,$$

$$\phi = \phi_{\mathrm{in}} \quad \text{on} \quad \Gamma_{\mathrm{in}}\,, \quad \text{and} \quad \boldsymbol{\vartheta} \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma \backslash \Gamma_{\mathrm{in}}\,,$$