Rockafellian Relaxation for PDE-Constrained Optimization under Uncertainty in the context of Risk Measures

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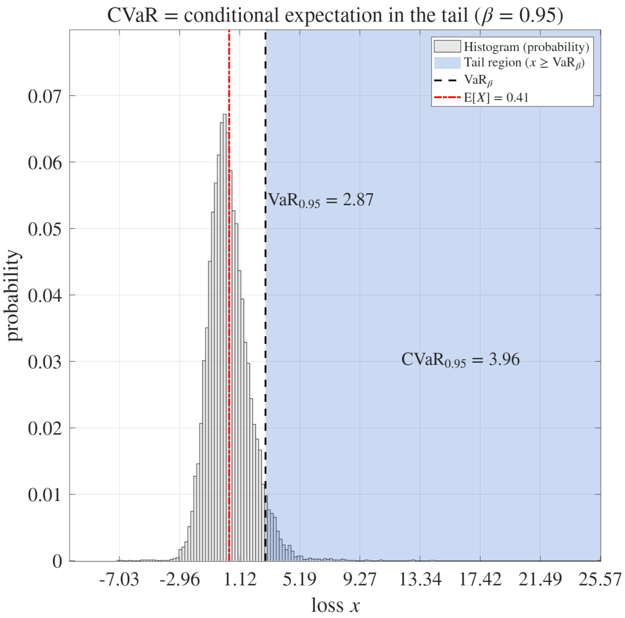
Motivation: Risk measures

Conditional Value-at-Risk

Given $\beta \in (0, 1)$ and a random variable X,

$$\mathrm{CVaR}_\beta(X) := \inf_{\gamma \in \mathbb{R}} \left\{ \gamma + \frac{1}{1-\beta} \mathbb{E}\big[(X - \gamma)_+\big] \right\}$$



CVaR = conditional expectation in the tail ($\beta = 0.95$)
Histogram (probability)
Tail region ($x \geq \mathrm{VaR}_\beta$)
VaR_β
$\mathrm{E}[X] = 0.41$
$\mathrm{VaR}_{0.95} = 2.87$
$\mathrm{CVaR}_{0.95} = 3.96$
probability
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
-7.03
-2.96
1.12
5.19
9.27
13.34
17.42
21.49
25.57
loss x

Motivation

Conditional Value-at-Risk

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$$\mathrm{CVaR}_\beta(X) := \inf_{\gamma\in\mathbb{R}} \left\{ \gamma + \frac{1}{1-\beta}\,\mathbb{E}\big[(X-\gamma)_+\big] \right\}$$

Let us consider the stochastic program $\min\limits_{x\in[0,1]} \varphi(x)$, where

$$\varphi(x) := \mathrm{CVaR}_\beta(g(x,\cdot))\,, \qquad \text{with} \;\; g(x,\xi) := \frac{1-x}{2} + \xi\, x\,,$$

and $\mathbb{P}(\xi = 0) = 1$. Then, $\varphi(x) = \frac{1-x}{2}$, and, consequently, the minimizer is $x^\star = 1$.

Now, for some $\varepsilon \in (0, 1)$, we introduce the corrupted random variable ξ_ε, whose law is given by $p_{\varepsilon,1} := \mathbb{P}[\xi_\varepsilon = 0] = 1 - \varepsilon$ and $p_{\varepsilon,2} := \mathbb{P}[\xi_\varepsilon = 1/\varepsilon] = \varepsilon$. Then, assuming that $\varepsilon < 1 - \beta$, it can be computed

$$\varphi_\varepsilon(x) = \mathrm{CVaR}_\beta(g(x, \xi_\varepsilon)) = \frac{1}{2} + \frac{1+\beta}{2(1-\beta)}\, x\,,$$

so the minimizer is $x_\varepsilon^\star = 0$.

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so the minimizer is $x_\varepsilon^\star = 0$.

Although ξ_ε converges to ξ in distribution, the corrupted minimizer never approaches the minimizer of the uncorrupted problem.

Definition

For Banach spaces X and Y, a function $\varphi : X \to \overline{\mathbb{R}}$, and a generic optimization problem

$$\min_{x \in X} \ \varphi(x)\,,$$

a bivariate function $\Phi : X \times Y \to \overline{\mathbb{R}}$ is called a Rockafellian for the problem, anchored at $\overline{y} \in Y$, if

$$\Phi(x, \overline{y}) = \varphi(x) \qquad \forall\, x \in X\,.$$

In our example, we can define the corrupted Rockafellian $\Phi_\varepsilon : [0,1] \times \mathbb{R}^2 \to \overline{\mathbb{R}}$ by

$$\Phi_\varepsilon(x,t) := \mathrm{CVaR}_\beta(g(x, \xi_\varepsilon + T)) + \frac{\theta_\varepsilon}{2}\,\|t\|_2^2 + \imath_\Delta(p_\varepsilon + t)$$

where $\theta_\varepsilon > 0$ and $\Delta := \big\{(q_1, q_2) \in [0,1]^2 \ : \quad q_1 + q_2 = 1\big\}$.

Let us consider

- $(\mathrm{Z}, \|\cdot\|_{\mathrm{Z}})$ a reflexive Banach space.
- $(\mathrm{U}, \|\cdot\|_{\mathrm{U}})$ an arbitrary Banach space.
- $(\Xi, \mathcal{A}, \mathbb{P})$ a probability space whose sample space is equipped with the norm $\|\cdot\|_{\Xi}$.
- Z_{ad} a closed and convex subset of Z.

Assume that Ξ is embedded in a finite-dimensional space.

We are interested in the problem

$$\min_{z \in \mathrm{Z}_{\mathrm{ad}}} f_0(z) + \mathrm{CVaR}_{\beta}[(g \circ s)(\cdot, z)],$$

where $f_0 : \mathrm{Z} \to \overline{\mathbb{R}}$, $g : \mathrm{U} \to \overline{\mathbb{R}}$, and $s : \Xi \times \mathrm{Z} \to \mathrm{U}$.

Properties of the solution map s

1. $s(\cdot, z) : \Xi \to \mathrm{U}$ is $\mathcal{A}$-measurable, for every $z \in \mathrm{Z}$.
2. If both $\xi_\varepsilon \to \xi$ in Ξ and $z_\varepsilon \rightharpoonup z$ in Z as $\varepsilon \downarrow 0$, then

$$s(\xi_\varepsilon, z_\varepsilon) \rightharpoonup s(\xi, z) \quad \text{in} \quad \mathrm{U}\,.$$

Properties of f_0 and g

1. f_0 is proper: $f_0(z) > -\infty \quad \forall\, z \in \mathrm{Z}$ and $f_0(z) < +\infty$ for some $z \in \mathrm{Z}$.
2. Both f_0 and g are sequentially weakly lower semicontinuous:

$$z_\varepsilon \xrightharpoonup{\mathrm{Z}} z \implies \liminf_{\varepsilon \downarrow 0} f_0(z_\varepsilon) \geqslant f_0(z)\,,$$

$$u_\varepsilon \xrightharpoonup{\mathrm{U}} u \implies \liminf_{\varepsilon \downarrow 0} g(z_\varepsilon) \geqslant g(z)\,.$$

The minimization problem

$$\min_{z \in \mathrm{Z}_{\mathrm{ad}}} f_0(z) + \mathrm{CVaR}_\beta[(g \circ s)(\cdot, z)]$$

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Augmented problem

$$\min_{(z,\gamma) \in \mathrm{Z}_{\mathrm{ad}} \times \mathbb{R}} \varphi(z, \gamma) := f_0(z) + \gamma + \kappa\, \mathbb{E}\Big[\big((g \circ s)(\cdot, z) - \gamma\big)_+\Big],$$

where $\kappa := 1/(1-\beta)$.

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Proposition

If Z_{ad} is bounded in Z or f_0 is coercive, then the augmented problem has a minimizer.

Better notation

Augmented problem

$$\min_{(z,\gamma)\in Z_{\mathrm{ad}}\times\mathbb{R}} \varphi(z,\gamma) := f_0(z) + \gamma + \kappa\,\mathbb{E}\big[\big((g\circ s)(\cdot,z)-\gamma\big)_+\big]$$

- $\mathscr{J}(z,\gamma) := g(s(\cdot,z)) - \gamma$.
- For $\mathbf{t} : \Xi \to \Xi$, we define $\mathscr{J}(z,\gamma;\mathbf{t}) = \mathscr{J}(z,\gamma)\circ\mathbf{t}$ and $\mathscr{J}_+(z,\gamma;\mathbf{t}) = (\mathscr{J}(z,\gamma;\mathbf{t}))_+$.
- $f : Z\times\mathbb{R} \to \overline{\mathbb{R}}$ as $f(z,\gamma) := f_0(z) + \gamma$.

Under this notation, the objective functional may be rewritten as

$$\varphi(z,\gamma) = f(z,\gamma) + \kappa\,\mathbb{E}[\mathscr{J}_+(z,\gamma)]\,.$$

Suppose that

1. There exists another σ-finite measure μ defined on $(\Xi, \mathcal{A})$.
2. $\mathbb{P}$ is absolutely continuous with respect to μ.

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Let ρ be the Radon–Nikodym derivative $d\mathbb{P}/d\mu$ and let $\mathcal{I} : \mathrm{L}^1(\Xi) \to \mathbb{R}$ denote the integral $X \mapsto \int_\Xi X \, d\mu$.

$$\varphi(z, \gamma) = f(z, \gamma) + \kappa \mathbb{E}[\mathscr{J}_+(z, \gamma)] \iff \varphi(z, \gamma) = f(z, \gamma) + \kappa \mathcal{I}\left[\mathscr{J}_+(z, \gamma)\, \rho\right]$$

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Given $\mathrm{t} : \Xi \to \mathbb{R}$ and $\mathbf{t} : \Xi \to \Xi$, we denote the translations

$$\wp(\mathrm{t}) = \rho + \mathrm{t} \quad \text{and} \quad \omega(\mathbf{t}) = I + \mathbf{t}\,,$$

Then,

$$\varphi(z,\gamma) = f(z,\gamma) + \kappa \, \mathcal{I}\left[\mathscr{J}_+\big(z,\gamma;\omega(0)\big)\, \wp(0)\right].$$

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For $q \in (1,+\infty)$ and $q' \in (1,+\infty)$,

$$\mathrm{T} := \mathrm{L}^q(\Xi) \quad \text{and} \quad \mathbf{T} := \mathrm{L}^{q'}(\Xi;\Xi)\,.$$

Define the Rockafellian $\Phi : \mathrm{Z}\times\mathbb{R}\times\mathrm{T}\times\mathbf{T}\to\overline{\mathbb{R}}$ by

$$\Phi(z,\gamma;\mathrm{t},\mathbf{t}) := \begin{cases} f(z,\gamma) + \kappa\mathcal{I}\left[\mathscr{J}_+\big(z,\gamma;\omega(\mathbf{t})\big)\wp(\mathrm{t})\right], & \text{if } (\mathrm{t},\mathbf{t}) = (0,0)\,, \\ +\infty & \text{otherwise}\,, \end{cases}$$

which is clearly anchored at $(\mathrm{t},\mathbf{t}) = (0,0)$.

δ-smoothed uncorrupted problem

δ-smoothed objective functionals

$$\varphi^\delta(z,\gamma) = f(z) + \kappa \mathcal{I}\left[\mathscr{J}_{+,\delta}\big(z,\gamma;\omega(0)\big)\,\wp(0)\right].$$

δ-smoothed Rockafellians

$$\begin{aligned}
&\Phi^\delta(z,\gamma;\mathrm{t},\mathbf{t}) \\
&\quad := \begin{cases} f(z,\gamma) + \kappa \mathcal{I}\left[\mathscr{J}_{+,\delta}\big(z,\gamma;\omega(\mathbf{t})\big)\,\wp(\mathrm{t})\right], & \text{if } (\mathrm{t},\mathbf{t}) = 0\,, \\ +\infty\,, & \text{otherwise}\,. \end{cases}
\end{aligned}$$

Smoothed-corrupted problems

Define the space of probability densities

$$\mathrm{P} := \left\{ \varrho : \Xi \to \mathbb{R}_+ \;\middle|\; \varrho \in \mathrm{L}^\infty(\Xi) \quad \text{and} \quad \int_\Xi \varrho(\xi)\, d\mu(\xi) = 1 \right\} .$$

Notice that $\rho \in \mathrm{P}$.

Let us consider $\{\eta_\varepsilon\}_{\varepsilon>0} \subset \mathbf{T}$ and $\{\rho_\varepsilon\}_{\varepsilon>0} \subset \mathrm{P}$, which represent corruption maps and corrupted distributions, respectively.

Define $\wp_\varepsilon(\mathrm{t}) = \rho_\varepsilon + \mathrm{t}$ and $\omega_\varepsilon(\mathbf{t}) = \eta_\varepsilon + \mathbf{t}$, for $X : \Xi \to \mathbb{R}$, $\mathrm{t} : \Xi \to \mathbb{R}$ and $\mathbf{t} : \Xi \to \Xi$, respectively.

The smoothed-corrupted objective functional reads:

$$\varphi_\varepsilon^\delta(z, \gamma) = f(z, \gamma) + \kappa \mathcal{I}\big[\, \mathscr{J}_{+,\delta}(z, \gamma; \omega_\varepsilon(0))\, \wp_\varepsilon(0) \big] .$$

We aim to introduce Rockafellians, prove that the induced minimization problem admits a solution, derive the corresponding optimality conditions, and relate the solutions of the corrupted Rockafellian problem with those of the uncorrupted one through weak-strong Γ-convergence.

Rockafellians

Assume that there exists a reflexive Banach space W consisting of functions $\Xi \to \mathbb{R}$ such that W is compactly embedded into $\mathrm{L}^q(\Xi)$.

Likewise, let $\mathbf{W}$ be a reflexive Banach space of functions $\Xi \to \Xi$ such that $\mathbf{W}$ is compactly embedded into $\mathrm{L}^{q'}(\Xi;\Xi)$.

Define

$$\mathrm{T}_\varepsilon := \mathrm{W} \oplus \mathrm{span}\{\rho_\varepsilon\} \quad \text{and} \quad \mathbf{T}_\varepsilon := \mathbf{W} \oplus \mathrm{span}\{I, \eta_\varepsilon\}\,.$$

We consider a function $\mathcal{H} : \mathrm{T} \times \mathbf{T} \to \mathbb{R}$ that is weakly lower semicontinuous, coercive, satisfies $\mathcal{H}(0,0) = 0$, and enjoys the following property: if $\mathcal{H}(\mathrm{t}_k, \mathbf{t}_k) \xrightarrow{k} 0$, then $(\mathrm{t}_k, \mathbf{t}_k) \xrightarrow{k} (0,0)$ strongly.

Furthermore, we introduce a sequence $(\theta_\varepsilon)_{\varepsilon>0} \subset \mathbb{R}_+$ of penalty parameters that satisfies $\theta_\varepsilon \xrightarrow{\varepsilon\to 0} +\infty$.

Rockafellians

Recall that

$$\varphi_\varepsilon^\delta(z,\gamma) = f(z,\gamma) + \kappa\mathcal{I}\big[\mathscr{J}_{+,\delta}(z,\gamma;\omega_\varepsilon(0))\,\wp_\varepsilon(0)\big]\,.$$

The Rockafellians associated with $\varphi_\varepsilon^\delta$ are defined as $\Phi_\varepsilon^\delta : \mathrm{Z}\times\mathbb{R}\times\mathrm{T}\times\mathbf{T}\to\overline{\mathbb{R}}$,

$$\begin{aligned}\Phi_\varepsilon^\delta(z,\gamma;\mathrm{t},\mathbf{t}) = f(z,\gamma) &+ \kappa\mathcal{I}[\mathscr{J}_{+,\delta}(z,\gamma;\omega_\varepsilon(\mathbf{t}))\,\wp_\varepsilon(\mathrm{t})] \\ &+ \theta_\varepsilon\,\mathcal{H}(\mathrm{t},\mathbf{t}) + \imath_{\mathrm{T}_\varepsilon\cap(\mathrm{P}-\rho_\varepsilon)}(\mathrm{t}) + \imath_{\mathbf{T}_\varepsilon}(\mathbf{t})\,.\end{aligned}$$

Assume that $1/q + 1/q' = 1$ and that there is a function $\varrho : [0,+\infty)\to[0,+\infty)$ such that

$$\big|(g\circ s)(\xi,z)\big| \leqslant \varrho(\|z\|_{\mathrm{Z}})\,\|\xi\|_{\Xi} \qquad \forall(\xi,z)\in\Xi\times\mathrm{Z}\,.$$

Then, the integral term in Φ_ε^δ is finite.

Existence of a solution

Theorem

Fix $\varepsilon > 0$ and $\delta > 0$. Assume that f_0 is coercive or Z_{ad} is bounded. Then, there exists

$$(z^\star, \gamma^\star; \mathrm{t}^\star, \mathbf{t}^\star) \in \mathrm{Z}_{\mathrm{ad}} \times \mathbb{R} \times \mathrm{T}_\varepsilon \times \mathbf{T}_\varepsilon$$

such that

$$\Phi_\varepsilon^\delta(z^\star, \gamma^\star; \mathrm{t}^\star, \mathbf{t}^\star) \leqslant \Phi_\varepsilon^\delta(z, \gamma; \mathrm{t}, \mathbf{t}) \quad \forall\, (z, \gamma, \mathrm{t}, \mathbf{t}) \in \mathrm{Z}_{\mathrm{ad}} \times \mathbb{R} \times \mathrm{T} \times \mathbf{T}\,.$$

Proof of the existence result

Recall that

$$\Phi^\delta_\varepsilon(z,\gamma;\mathrm{t},\mathbf{t}) = f(z,\gamma) + \kappa\,\mathcal{I}[\mathscr{J}_{+,\delta}(z,\gamma;\omega_\varepsilon(\mathbf{t}))\,\wp_\varepsilon(\mathrm{t})] \\ + \theta_\varepsilon\,\mathcal{H}(\mathrm{t},\mathbf{t}) + \imath_{\mathrm{T}_\varepsilon\cap(\mathrm{P}-\rho_\varepsilon)}(\mathrm{t}) + \imath_{\mathbf{T}_\varepsilon}(\mathbf{t})\,.$$

Define the auxiliary functional $\widetilde{\Phi}^\delta_\varepsilon : \mathrm{Z}\times\mathbb{R}\times\mathrm{T}_\varepsilon\times\mathbf{T}_\varepsilon \to \overline{\mathbb{R}}$ given by

$$\widetilde{\Phi}^\delta_\varepsilon(z,\gamma;\mathrm{t},\mathbf{t}) := \Phi^\delta_\varepsilon|_{\mathrm{Z}\times\mathbb{R}\times\mathrm{T}_\varepsilon\times\mathbf{T}_\varepsilon}(z,\gamma;\mathrm{t},\mathbf{t}) + \imath_{\mathrm{Z}_{\mathrm{ad}}}(z)\,,$$

If $\widetilde{\Phi}^\delta_\varepsilon$ admits a minimizer, then any such minimizer is also a minimizer of Φ^δ_ε.

Proof of the existence result: $\widetilde{\Phi}_\varepsilon^\delta$ is s.w.l.s.

First step: $\widetilde{\Phi}_\varepsilon^\delta$ is sequentially weakly lower semicontinuous.

Consider sequences $\{z_n\}_{n\in\mathbb{N}} \subset \mathrm{Z}$, $\{\gamma_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, $\{\mathrm{t}_n\}_{n\in\mathbb{N}} \subset \mathrm{T}_\varepsilon$ and $\{\mathbf{t}_n\}_{n\in\mathbb{N}} \subset \mathbf{T}_\varepsilon$ converging weakly to $z \in \mathrm{Z}$, $\gamma \in \mathbb{R}$, $\mathrm{t} \in \mathrm{T}_\varepsilon$ and $\mathbf{t} \in \mathbf{T}_\varepsilon$, respectively.

- Case 1: If $(z_n, \mathrm{t}_n)_{n\in\mathbb{N}}$ does not have a subsequence contained in $\mathrm{Z}_{\mathrm{ad}} \times (\mathrm{P} - \rho_\varepsilon)$. We are done!
- Case 2: There exists a (not relabeled) subsequence such that $z_n \in \mathrm{Z}_{\mathrm{ad}}$ and $\rho_\varepsilon + \mathrm{t}_n \in \mathrm{P}$, for all $n \in \mathbb{N}$. We may assume that the limit inferior is attained along this subsequence.

Since $\mathrm{t}_n \rightharpoonup \mathrm{t}$ in $\mathrm{T}_\varepsilon = \mathrm{W} \oplus \mathrm{span}\{\rho_\varepsilon\} \overset{c}{\hookrightarrow} \mathrm{L}^q(\Xi) \oplus \mathrm{span}\{\rho_\varepsilon\}$, then

$$\underbrace{\rho_\varepsilon(\xi) + \mathrm{t}_{n'}(\xi)}_{=\wp_\varepsilon(\mathrm{t}_{n'})} \xrightarrow{n' \to \infty} \underbrace{\rho_\varepsilon(\xi) + \mathrm{t}(\xi)}_{=\wp_\varepsilon(\mathrm{t})} \quad \text{for a.e. } \ \xi \in \Xi\,.$$

Similarly,

$$\eta_\varepsilon(\xi) + \mathbf{t}_{n'}(\xi) \xrightarrow{n' \to \infty} \eta_\varepsilon(\xi) + \mathbf{t}(\xi) \quad \text{for a.e. } \ \xi \in \Xi\,,$$

which combined with the fact that $z_n \rightharpoonup z$ in Z and the assumption on s,

$$s(\eta_\varepsilon(\xi) + \mathbf{t}_{n'}(\xi), z_{n'}) \rightharpoonup s(\eta_\varepsilon(\xi) + \mathbf{t}(\xi), z) \quad \text{ in } \ \mathrm{U}\,, \quad \text{for a.e. } \ \xi \in \Xi\,.$$

Since g is s.w.l.s., the above convergence implies that

$$\liminf_{n' \to \infty} \ \Big((g \circ s)(\eta_\varepsilon(\xi) + \mathbf{t}_{n'}(\xi), z_{n'})\Big) \geqslant (g \circ s)(\eta_\varepsilon(\xi) + \mathbf{t}(\xi), z)$$

for a.e. $\xi \in \Xi$.

Using that $\gamma_{n'} \to \gamma$ in $\mathbb{R}$, the monotonicity and continuity of $(\,\cdot\,)_{+,\delta}$, and the foregoing inequality, we arrive at

$$\liminf_{n' \to \infty} \big(\mathscr{I}_{+,\delta}(z_{n'}, \gamma_{n'}; \omega_\varepsilon(\mathbf{t}_{n'}))\big)(\xi) \geqslant \big(\mathscr{I}_{+,\delta}(z, \gamma; \omega_\varepsilon(\mathbf{t}))\big)(\xi) \quad \text{a.e. in} \quad \Xi\,.$$

Since $\wp_\varepsilon(\mathrm{t}_{n'}) \to \wp_\varepsilon(t)$ pointwise in Ξ and they are non-negative,

$$\liminf_{n' \to \infty} \mathscr{I}_{+,\delta}(z_{n'}, \gamma_{n'}; \omega_\varepsilon(\mathbf{t}_{n'}))\, \wp_\varepsilon(\mathrm{t}_{n'}) \geqslant \mathscr{I}_{+,\delta}(z, \gamma; \omega_\varepsilon(\mathbf{t}))\, \wp_\varepsilon(\mathrm{t}) \quad \text{a.e. in} \quad \Xi\,.$$

By Fatou's Lemma,

$$\liminf_{n' \to \infty} \mathcal{I}\Big(\mathscr{I}_{+,\delta}\big(z_{n'}, \gamma_{n'}; \omega_\varepsilon(\mathbf{t}_{n'})\big)\, \wp_\varepsilon(\mathrm{t}_{n'})\Big) \geqslant \mathcal{I}\Big(\mathscr{I}_{+,\delta}\big(z, \gamma; \omega_\varepsilon(\mathbf{t})\big)\, \wp_\varepsilon(\mathrm{t})\Big)\,.$$

Recalling that f_0 and $\mathcal{H}$ are w.l.s.

$$\liminf_{n\to\infty} \widetilde{\Phi}_\varepsilon^\delta(z_n,\gamma_n;\mathrm{t}_n,\mathbf{t}_n) = \liminf_{n'\to\infty} \widetilde{\Phi}_\varepsilon^\delta(z_{n'},\gamma_{n'};\mathrm{t}_{n'},\mathbf{t}_{n'}) \geqslant \widetilde{\Phi}_\varepsilon^\delta(z,\gamma;\mathrm{t},\mathbf{t})$$

Therefore, $\widetilde{\Phi}_\varepsilon^\delta$ is s.w.l.s.

Second step: $\widetilde{\Phi}_\varepsilon^\star$ is coercive.

It follows from the fact that $\mathcal{H}$ is coercive, and that f is coercive or Z_{ad} is bounded.

Conclusion!

Since $\mathrm{Z} \times \mathbb{R} \times \mathrm{T}_\varepsilon \times \mathbf{T}_\varepsilon$ is a reflexive Banach space, $\widetilde{\Phi}_\varepsilon^\star$ is s.w.l.s. and coercive, we are able to use the Weierstrass minimization theorem to conclude.

$\square$

Optimality conditions

Let $\left(z_\varepsilon^\star, \gamma_\varepsilon^\star; \mathrm{t}_\varepsilon^\star, \mathbf{t}_\varepsilon^\star\right) \in \mathrm{Z}_{\mathrm{ad}} \times \mathbb{R} \times \mathrm{T}_\varepsilon \cap (\mathrm{P} - \rho_\varepsilon) \times \mathbf{T}_\varepsilon$ be an optimal solution of the Rockafellian problem, and set $\lambda_\varepsilon^\star := (\omega_\varepsilon(\mathbf{t}_\varepsilon^\star), z_\varepsilon^\star)$. Then,

1. For all $z \in \mathrm{Z}_{\mathrm{ad}}$,

$$\left\langle \kappa \mathcal{I}\big[\wp_\varepsilon(t_\varepsilon^\star)\, A_\delta\big(\mathscr{J}(z_\varepsilon^\star, \gamma_\varepsilon^\star; \omega_\varepsilon(\mathbf{t}_\varepsilon^\star))\big)\, \mathbf{D}_z s(\lambda_\varepsilon^\star)^* \,\mathbf{D} g(s(\lambda_\varepsilon^\star))\big] + \mathbf{D} f_0(z_\varepsilon^\star)\,,\, z - z_\varepsilon^\star \right\rangle \geqslant 0\,.$$

2. It holds

$$\mathcal{I}\Big[\wp_\varepsilon(\mathrm{t}_\varepsilon^\star)\, A_\delta\big(\mathscr{J}(z_\varepsilon^\star, \gamma_\varepsilon^\star; \omega_\varepsilon(\mathbf{t}_\varepsilon^\star))\big)\Big] = \frac{1}{\kappa}\,.$$

3. For all $\mathrm{t} \in \mathrm{T}_\varepsilon \cap (\mathrm{P} - \rho_\varepsilon)$,

$$\left\langle \theta_\varepsilon\, \mathbf{D}_{\mathrm{t}} \mathcal{H}(\mathrm{t}_\varepsilon^\star, \mathbf{t}_\varepsilon^\star) + \kappa \mathcal{I}\Big[\mathscr{J}_{+,\delta}(z_\varepsilon^\star, \gamma_\varepsilon^\star; \omega_\varepsilon(\mathbf{t}_\varepsilon^\star))\, I\Big], \mathrm{t} - \mathrm{t}_\varepsilon^\star \right\rangle \geqslant 0\,.$$

4. For all $\mathbf{t} \in \mathbf{T}_\varepsilon$,

$$\left\langle \kappa \mathcal{I}\Big[\wp_\varepsilon(\mathrm{t}_\varepsilon^\star)\, A_\delta\big(\mathscr{J}(z_\varepsilon^\star, \gamma_\varepsilon^\star; \omega_\varepsilon(\mathbf{t}_\varepsilon^\star))\big)\, \mathbf{D}_{\mathbf{t}} s(\lambda_\varepsilon^\star)^* \,\mathbf{D} g(s(\lambda_\varepsilon^\star)\Big] + \theta_\varepsilon\, \mathbf{D}_{\mathbf{t}} \mathcal{H}(\mathrm{t}_\varepsilon^\star, \mathbf{t}_\varepsilon^\star), \mathbf{t} - \mathbf{t}_\varepsilon^\star \right\rangle \geqslant 0\,.$$

Weak-strong Γ-convergence

Let X and Y be Banach spaces. Consider a map $\phi : X \times Y \to \mathbb{R}$ and a sequence $(\phi_\varepsilon)_\varepsilon$ of maps from $X \times Y$ to $\mathbb{R}$.

Definition

The sequence $(\phi_\varepsilon)_\varepsilon$ weak-strong Γ-converges to ϕ, denoted by $\phi_\varepsilon \xrightarrow{\Gamma} \phi$, if the following conditions hold:

1. For all $(x, y) \in X \times Y$, there exists a sequence $(x_\varepsilon, y_\varepsilon)_{\varepsilon>0}$ in $X \times Y$ converging strongly to (x, y) such that
$$\limsup_{\varepsilon \downarrow 0} \phi_\varepsilon(x_\varepsilon, y_\varepsilon) \leqslant \phi(x, y) .$$
2. For all sequences $x_\varepsilon \rightharpoonup x \in X$ and $y_\varepsilon \to y \in Y$, there holds
$$\liminf_{\varepsilon \downarrow 0} \phi_\varepsilon(x_\varepsilon, y_\varepsilon) \geqslant \phi(x, y) .$$

Proposition

Let $(x_\varepsilon^\star, y_\varepsilon^\star)_\varepsilon$ be a sequence in $X \times Y$ with $(x_\varepsilon^\star, y_\varepsilon^\star) \in \operatorname{argmin} \phi_\varepsilon(x, y)$. Assume that $\phi_\varepsilon \xrightarrow{\Gamma} \phi$, $x_\varepsilon^\star \rightharpoonup x^\star$ and $y_\varepsilon^\star \to y^\star$ as $\varepsilon \downarrow 0$. Then, $(x^\star, y^\star) \in \operatorname{argmin} \phi(x, y)$.

Theorem (Weak-strong Γ-convergence)

Let $\rho \in \mathrm{P} \cap \mathrm{T}_\varepsilon$ for all $\varepsilon > 0$ and consider corruptions $(\rho_\varepsilon)_{\varepsilon>0}$ and $(\eta_\varepsilon)_{\varepsilon>0}$ such that $\rho_\varepsilon \in \mathrm{P} \cap \mathrm{T}_\varepsilon$ and $\eta_\varepsilon \in \mathbf{T} \cap \mathbf{T}_\varepsilon$, for each $\varepsilon > 0$. Assume that

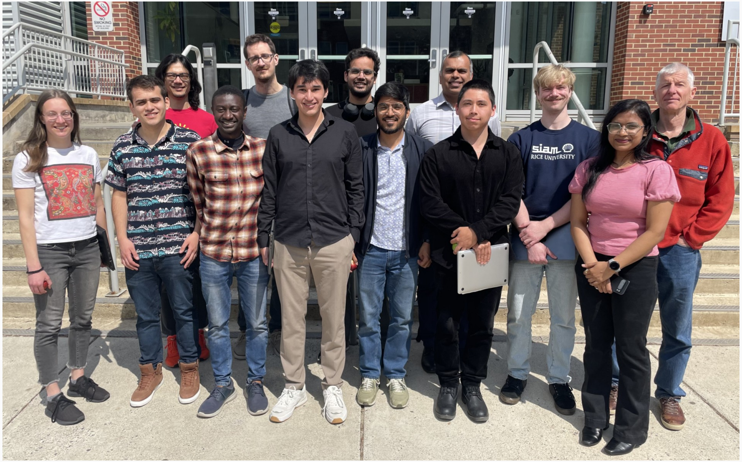
$$\lim_{\varepsilon \downarrow 0} \theta_\varepsilon = +\infty \,,$$

and

$$\lim_{\varepsilon \downarrow 0} \theta_\varepsilon \, \mathcal{H}(\rho - \rho_\varepsilon, \eta_\varepsilon - I) = 0 \,.$$

Then, $\Phi_\varepsilon^\delta \xrightarrow{\Gamma} \Phi^\delta$ as $\varepsilon \downarrow 0$.

Thanks!!!



With the GMU crew

How to deal with the positive part in the CVaR?

Let $\zeta : \mathbb{R} \to \mathbb{R}$ be a function satisfying

1. ζ is continuous and bounded on the real line.
2. ζ is nonnegative and $\int_{\mathbb{R}} \zeta(x)\, dx = 1$.
3. $\int_{\mathbb{R}} \zeta(x)|x|\, dx < +\infty$.
4. Either $\int_{\mathbb{R}} \zeta(x)x\, dx \leqslant 0$ or $\int_{-\infty}^{0} \zeta(x)|x|\, dx = 0$.
5. The set $\{x \in \mathbb{R} \;:\; \zeta(x) > 0\}$ is connected.

We define a net of smoothings of the positive part $(\,\cdot\,)_{+,\delta} : \mathbb{R} \to \mathbb{R}$, given by

$$(x)_{+,\delta} := \int_{-\infty}^{x} A_\delta(\tau)\, d\tau, \quad \text{where} \quad A_\delta(\tau) := \int_{-\infty}^{\tau} \frac{1}{\delta}\zeta\left(\frac{\sigma}{\delta}\right) d\sigma\,,$$

for all $\delta > 0$.